Scientific activity report for the research grant

Variational approaches to set-valued optimization and applications

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The scientific activity in the period January 2018 – December 2018 was focused on the objectives of the project associated to this stage. In our investigations we obtained several results within the proposed objectives and these results are contained in the papers [6], [13], [8], [2] (two published and two submitted for publication). For more details, see

$$http://www.math.uaic.ro/\tilde{}zalinesc/ID-0188-ro.html$$

We briefly present next the main ideas and assertions from the mentioned articles.

In the paper [6] we introduced a notion of Henig proper efficiency for constrained vector optimization problems in the setting of variable ordering structure. In order to get an appropriate concept, we had to explore firstly the case of fixed ordering structure and to observe that, in certain situations, the well-known Henig proper efficiency can be expressed in a simpler way. Then, we observe that the newly introduced notion can be reduced, by a Clarke-type penalization result, to the notion of unconstrained robust efficiency. We show that this penalization technique, coupled with sufficient conditions for weak openness, serves as a basis for developing necessary optimality conditions for our Henig proper efficiency in terms of generalized differentiation objects lying in both primal and dual spaces.

More precisely, the setting of this work is as follows. Let X, Y be normed vector spaces. For $x \in X$ and $r > 0$, denote by $B_X(x,r)$, $D_X(x,r)$ and $S_X(x,r)$ the open and the closed balls, and the sphere of center x and radius r, respectively. In the case where $x := 0$ and $r := 1$, we use the notations B_X, D_X and S_X. For $x \in X$, the symbol $\mathcal{V}(x)$ stands for the system of neighborhoods of x. For a set $A \subset X$, we denote by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$ its topological interior, closure and boundary, respectively. The cone generated by A is designated by $\operatorname{cone} A$, and the convex hull of A is $\operatorname{conv} A$. The distance from a point x to a set A is $d(x, A) := \inf\{\|x - a\| \mid a \in A\}$. The notation X^* stands for the topological dual of X. On a product space we consider the sum norm, unless otherwise stated.

Let $F : X \rightrightarrows Y$ be a multifunction. The graph of F is denoted by $\operatorname{Gr} F = \{(x,y) \in X \times Y \mid y \in F(x)\}$. If $A \subset X$, then $F(A) := \bigcup_{x \in A} F(x)$ and the inverse set-valued mapping of F is $F^{-1} : Y \rightrightarrows X$ given by $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$.

Consider $\emptyset \neq M \subset X$ as a closed set and $C \subset Y$ as a closed convex pointed cone. We recall that a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$ is an efficient point for F on M with respect to C if there is a neighborhood U of $\overline{x}$ such that for every $x \in M \cap U$ one has

$$(F(x) - \overline{y}) \cap (-C) \subset \{0\}.$$

Denote by $\operatorname{Eff}(F, M; C)$ the set of efficient points for F on M with respect to C.

If $\operatorname{int} C \neq \emptyset$, a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$ is a weakly efficient point for F on M with respect to C if there is a neighborhood U of $\overline{x}$ such that for every $x \in M \cap U$ one has

$$(F(x) - \overline{y}) \cap (-\operatorname{int} C) = \emptyset,$$

and we denote by $\operatorname{WEff}(F, M; C)$ the set of weakly efficient points for F on M with respect to C.

Now, following [14, p. 110], the set of Henig-proper efficient points of F on M with respect to C is

$$\operatorname{HEff}(F, M; C) = \bigcup\{\operatorname{Eff}(F, M; K) \mid K \text{ convex cone, } C \setminus \{0\} \subset \operatorname{int} K \neq Y\}.$$

When $M = X$, then we have unconstrained efficiencies and we drop the notation M in the writing of efficiency sets.

A convex set B is said to be a base for the cone C if $0 \notin \operatorname{cl} B$ and $C = \operatorname{cone} B$. A cone which admits a base is called based. In this case one can consider the so-called Henig dilating cones C_ε, $\varepsilon \in (0, d(0, B))$ for which the definition and properties are presented, for instance, in [14, Lemma 3.2.51].

The next cone separation result is obtained in [5].

Theorem 1 *Let $P, Q \subset Y$ be closed cones such that Q admits a closed base B. Moreover, suppose that P or B is a.c. If $P \cap Q = \{0\}$, then there exists $\varepsilon \in]0, d(0,B)[$ such that $P \cap Q_\varepsilon = \{0\}$.*

Now, if one supposes that C has a closed asymptotically compact (a.c., for short) base B and take, as above, $\delta = d(0, B)$, then we can write the following relation:

$$\text{HEff}(F, M; C) = \bigcup \{\text{Eff}(F, M; C_\varepsilon) \mid \varepsilon \in]0, \delta[\}.$$

This remark is essential in what follows.

Consider $K : X \rightrightarrows Y$ a multifunction whose values are proper pointed closed convex cones. This leads us, for every $x \in X$, to an order relation on $Y : y_1 \leq_{K(x)} y_2 \Leftrightarrow y_2 - y_1 \in K(x)$. Suppose that all cones $K(x)$ are based. On the basis of the previous discussion, we define the set of Henig-proper nondominated points of F on M with respect to K as

$$\begin{aligned} \text{VosHEff}(F, M; K) = \{(\overline{x}, \overline{y}) \in \text{Gr}\, F \cap (M \times Y) \mid \exists \varepsilon > 0, \exists U \in \mathcal{V}(\overline{x}), \forall x \in U \cap M, \\ (F(x) - \overline{y}) \cap (-K(x)_\varepsilon) \subset \{0\}\}. \end{aligned} \tag{1}$$

This newly introduced concept clearly covers the case of Henig proper efficiency described before provided that the base of $K(x)$ with x close to $\overline{x}$ is a.c., and, moreover, this is in the same line with other notions of efficiency developed in [11] and [12]. We give a short account on these concepts, since we investigate in the sequel some interesting links between all these efficiencies.

According to [11], a point $(\overline{x}, \overline{y}) \in \text{Gr}\, F \cap (M \times Y)$ is a (local) nondominated point for F on M with respect to K if there is a neighborhood U of $\overline{x}$ such that for every $x \in M \cap U$, one has

$$(F(x) - \overline{y}) \cap (-K(x)) \subset \{0\}.$$

If $\text{int}\, K(x) \neq \emptyset$ for every x in a neighborhood V of $\overline{x}$, a point $(\overline{x}, \overline{y}) \in \text{Gr}\, F \cap (M \times Y)$ is a weakly nondominated point for F on M with respect to K if there is a neighborhood $U \subset V$ of $\overline{x}$ such that for every $x \in M \cap U$, one has

$$(F(x) - \overline{y}) \cap (-\,\text{int}\, K(x)) = \emptyset.$$

As a natural generalization of the above notions, the following concepts of robustness were introduced in [12].

A point $(\overline{x}, \overline{y}) \in \text{Gr}\, F \cap (M \times Y)$ is a local robust efficient point for F on M with respect to K if there is a neighborhood U of $\overline{x}$ such that for every $x, z \in U \cap M$, one has

$$(F(x) - \overline{y}) \cap (-K(z)) \subset \{0\}.$$

If $\text{int}\, K(z) \neq \emptyset$ for every z in a neighborhood V of $\overline{x}$, then one says that a point $(\overline{x}, \overline{y}) \in \text{Gr}\, F \cap (M \times Y)$ is a local robust weakly efficient point for F on M with respect to K if there is a neighborhood $U \subset V$ of $\overline{x}$ such that for every $x, z \in U \cap M$, one has

$$(F(x) - \overline{y}) \cap (-\,\text{int}\, K(z)) = \emptyset.$$

When $M = X$, we have unconstrained efficiencies and we omit to write "on M".

In order to investigate the constrained Henig-proper nondomination from the perspective of necessary optimality conditions, we firstly need to reduce this constrained efficiency to an unconstrained one. As usual, this can be done by a penalization result and this task is accomplished in the next theorem which is in the line of Clarke penalization method, that is, asks for some local Lipschitz behavior of the involved multifunctions and uses the distance to the set of feasible points M as a penalty term. Moreover, we prove that a $\text{VosHEff}(F, M; K)$ point is reduced in this way to an unconstrained robust efficient point.

In the first penalization result, we consider the situation when the ordering cones have nonempty topological interior, that is the setting known in vector optimization as the weak case.

Theorem 2 *Let $(\overline{x}, \overline{y}) \in \text{Gr}\, F \cap (M \times Y)$. Suppose that there exists a neighborhood U of $\overline{x}$ such that:*

(i) $\text{int}\, K(x) \neq \emptyset$ for all $x \in U$ and there exists $e \in Y \setminus \{0\}$ such that $e \in K(x)$ for all $x \in U$;

(ii) there exists a constant $L > 0$ such that for all $u, v \in U$,

$$F(u) \subset F(v) - L \|u - v\| e + K(v);$$

(iii) $K(x)$ *is based for all* $x \in U$ *(the base of* $K(x)$ *is denoted by* $B(x)$*);*
(iv) there exists a constant $l > 0$ *such that for all* $u, v \in U$,

$$B(u) \subset B(v) + l \|u - v\| D_Y.$$

If $(\overline{x}, \overline{y}) \in \operatorname{VosHEff}(F, M; K)$, *then* $(\overline{x}, \overline{y})$ *is a local robust weakly efficient point for* $F(\cdot) + Ld_M(\cdot)e$ *with respect to* K.

Now we do not suppose that $\operatorname{int} K(x) \neq \emptyset$ and then we get a result for the strong case.

Theorem 3 *Let* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$. *Suppose that there exists a neighborhood* U *of* $\overline{x}$ *such that:*
(i) there exists $e \in Y \setminus \{0\}$ *such that* $e \in K(x)$ *for all* $x \in U$;
(ii) there exists a constant $L > 0$ *such that for all* $u, v \in U$,

$$F(u) \subset F(v) - L \|u - v\| e + K(v);$$

(iii) $K(x)$ *is based for all* $x \in U$ *(the base of* $K(x)$ *is denoted by* $B(x)$*);*
(iv) there exists a constant $l > 0$ *such that for all* $u, v \in U$,

$$B(u) \subset B(v) + l \|u - v\| D_Y.$$

If $(\overline{x}, \overline{y}) \in \operatorname{VosHEff}(F, M; K)$, *then for all* $\beta > L$, $(\overline{x}, \overline{y})$ *is a local robust efficient point for* $F(\cdot) + \beta d_M(\cdot)e$ *with respect to* K.

Let $F,\ K : X \rightrightarrows Y$ be the multifunctions considered previously. We aim to get necessary optimality conditions for Henig-proper nondominated points and in order to do this, according to the penalty results from the previous section (that are, Theorems 2 and 3), we need to provide appropriate necessary optimality conditions for the robust efficiency for the sum of two mappings (the objective F and the penalty term).

Recall (see [12]) that the pair (F, K) is weakly open at $(\overline{x}, \overline{y})$, or $F+K$ is weakly open at $(\overline{x}, \overline{y}) \in \operatorname{Gr}(F+K)$ if for every neighborhood U of $\overline{x}$, there exists a neighborhood V of $\overline{y}$ such that $V \subset F(U) + K(U)$. Motivated by the incompatibility proved in [12] between weak openness and the robust efficiency, and by the fact that the penalty results presented before involve three mappings, in this section we obtain sufficient conditions for the weak openness of pairs of the form $(F + G, K)$, where $G : X \rightrightarrows Y$ is a multifunction.

We denote by (F, G) the multifunction $(F, G) : X \rightrightarrows Y \times Y$ given by $(F, G)(x) := F(x) \times G(x)$ for every $x \in X$, and by (F, G, K) the multifunction $(F, G, K) : X \rightrightarrows Y \times Y \times Y$ given by $(F, G, K)(x) := F(x) \times G(x) \times K(x)$ for every $x \in X$.

Recall the main generalized differentiation objects defined on primal spaces. The definitions are standard (see, for instance, [1]).

Definition 4 *Let* S *be a nonempty subset of* X *and* $\overline{x} \in X$.
(i) The Bouligand tangent cone to S *at* $\overline{x}$ *is the set*

$$T_B(S, \overline{x}) = \{u \in X \mid \exists (t_n) \downarrow 0, \exists (u_n) \to u, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \overline{x} + t_n u_n \in S\},$$

where $(t_n) \downarrow 0$ *means* $(t_n) \subset]0, \infty[$ *and* $(t_n) \to 0$.
(ii) The Ursescu tangent cone to S *at* $\overline{x}$ *is the set*

$$T_U(S, \overline{x}) = \{u \in X \mid \forall (t_n) \downarrow 0, \exists (u_n) \to u, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \overline{x} + t_n u_n \in S\}.$$

Based on these concepts, one defines the associated derivatives for set-valued maps.

Definition 5 *Let* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. *The Bouligand derivative of* F *at* $(\overline{x}, \overline{y})$ *is the set valued map* $D_B F(\overline{x}, \overline{y})$ *from* X *into* Y *defined by*

$$\operatorname{Gr} D_B F(\overline{x}, \overline{y}) = T_B(\operatorname{Gr} F, (\overline{x}, \overline{y})).$$

Note that the Ursescu derivative, denoted by $D_U F(\overline{x}, \overline{y})$, has a similar definition. One says that a set-valued map F is proto-differentiable at $\overline{x}$ relative to $\overline{y} \in F(\overline{x})$ if $D_U F(\overline{x}, \overline{y}) = D_B F(\overline{x}, \overline{y})$ (see [18]). Finally, the set-valued mapping F is said to have the Aubin property around $(\overline{x}, \overline{y})$ with constant $M > 0$ if there exist a neighborhood U of $\overline{x}$ and a neighborhood V of $\overline{y}$ such that, for every $x, u \in U$,

$$F(x) \cap V \subset F(u) + M \|x - u\| D_Y.$$

The first weak openness result, with the assumption in terms of derivatives, is given in the next theorem.

Theorem 6 *Let X, Y be Banach spaces, F, G, $K : X \rightrightarrows Y$ be multifunctions such that $\operatorname{Gr} F$, $\operatorname{Gr} G$, $\operatorname{Gr} K$ are closed around $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, $(\overline{x}, \overline{w}) \in \operatorname{Gr} G$ and $(\overline{x}, \overline{z}) \in \operatorname{Gr} K$, respectively. Suppose, moreover, that there exists $\lambda > 0$ such that, for every $(x, y, w, t, z) \in \operatorname{Gr}(F, G) \times \operatorname{Gr} K$ around $(\overline{x}, \overline{y}, \overline{w}, \overline{x}, \overline{z})$, the following assumptions are satisfied:*

(i) the next inclusion holds

$$B_Y(0,\lambda) \subset \operatorname{cl}[(D_B F(x,y) + D_B G(x,w) \cap B_Y(0,1))\,(B_X(0,1)) + D_B K(t,z)\,(B_X(0,1)) \cap B_Y(0,1)]; \quad (2)$$

(ii) either G is proto-differentiable at x relative to w and K is proto-differentiable at t relative to z, or G is proto-differentiable at x relative to w and F is proto-differentiable at x relative to y, or F is proto-differentiable at x relative to y and K is proto-differentiable at t relative to z;

(iii) either F has the Aubin property around the point (x, y), or G has the Aubin property around the point (x, w).

Then there exists $\varepsilon > 0$, such that, for every $(x, y, w, z) \in \operatorname{Gr}(F, G, K)$ around $(\overline{x}, \overline{y}, \overline{w}, \overline{z})$, and for every $\rho \in]0, \varepsilon[$,

$$B_Y(y + w + z, \lambda\rho) \subset (F + G)\,(B_X(x,\rho)) + K\,(B_X(x,\rho)),$$

and, consequently, $(F + G, K)$ is weakly open at $(\overline{x}, \overline{y} + \overline{z} + \overline{w})$.

Remark that if in the previous theorem we take $K(x) := 0$ for every $x \in X$, then we obtain Theorem 4.4 from [11], which is an openness result for $F + G$. Furthermore, in case $G(x) := 0$ for every $x \in X$, then we obtain the following weak openness result.

Theorem 7 *Let X, Y be Banach spaces, F, $K : X \rightrightarrows Y$ be multifunctions such that $\operatorname{Gr} F$, $\operatorname{Gr} K$ are closed around $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ and $(\overline{x}, \overline{z}) \in \operatorname{Gr} K$, respectively. Suppose, moreover, that there exists $\lambda > 0$ such that, for every $(x, y, t, z) \in \operatorname{Gr} F \times \operatorname{Gr} K$ around $(\overline{x}, \overline{y}, \overline{x}, \overline{z})$, the following assumptions are satisfied:*

(i) the next inclusion holds

$$B_Y(0,\lambda) \subset \operatorname{cl}[D_B F(x,y)\,(B_X(0,1)) + D_B K(t,z)\,(B_X(0,1)) \cap B_Y(0,1)]; \quad (3)$$

(ii) either F is proto-differentiable at x relative to y, or K is proto-differentiable at t relative to z.

Then there exists $\varepsilon > 0$, such that, for every $(x, y, z) \in \operatorname{Gr}(F, K)$ around $(\overline{x}, \overline{y}, \overline{z})$, and for every $\rho \in]0, \varepsilon[$,

$$B_Y(y + z, \lambda\rho) \subset F(B_X(x,\rho)) + K\,(B_X(x,\rho)), \quad (4)$$

and, consequently, (F, K) is weakly open at $(\overline{x}, \overline{y} + \overline{z})$.

Now consider the same type of results, but using the generalized differentiation objects lying in dual spaces. On the dual spaces, we work with the constructions developed by Mordukhovich and his collaborators (see [15]). Some of these concepts are briefly listed here.

Let X be a normed vector space, S be a non-empty subset of X and let $x \in S$, $\varepsilon \geq 0$. The set of $\varepsilon-$normals to S at x is

$$\widehat{N}_\varepsilon(S,x) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{S} x} \frac{x^*(u - x)}{\|u - x\|} \leq \varepsilon \right\}, \quad (5)$$

where $u \xrightarrow{S} x$ means that $u \to x$ and $u \in S$.

If $\varepsilon = 0$, the elements in the right-hand side of (5) are called Fréchet normals and their collection, denoted by $\widehat{N}(S, x)$, is the Fréchet normal cone to S at x.

Let $\overline{x} \in S$. The basic (or limiting, or Mordukhovich) normal cone to S at $\overline{x}$ is

$$N(S,\overline{x}) := \{x^* \in X^* \mid \exists \varepsilon_n \downarrow 0, x_n \xrightarrow{S} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}_{\varepsilon_n}(S, x_n), \forall n \in \mathbb{N}\}.$$

If X is an Asplund space, and S is closed around $\overline{x}$, the formula for the basic normal cone looks as follows:

$$N(S,\overline{x}) = \{x^* \in X^* \mid \exists x_n \xrightarrow{S} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}(S, x_n), \forall n \in \mathbb{N}\}. \quad (6)$$

Accordingly, two concepts of coderivatives for set-valued maps are in order.

Let $F : X \rightrightarrows Y$ be a set-valued map and $(\overline{x},\overline{y}) \in \operatorname{Gr} F$. Then the Fréchet coderivative of F at $(\overline{x},\overline{y})$ is the set-valued map $\widehat{D}^*F(\overline{x},\overline{y}) : Y^* \rightrightarrows X^*$ given by

$$\widehat{D}^*F(\overline{x},\overline{y})(y^*) := \{x^* \in X^* \mid (x^*,-y^*) \in \widehat{N}(\operatorname{Gr} F,(\overline{x},\overline{y}))\}.$$

Similarly, the normal coderivative of F at $(\overline{x},\overline{y})$ is the set-valued map $D^*F(\overline{x},\overline{y}) : Y^* \rightrightarrows X^*$ given by

$$D^*F(\overline{x},\overline{y})(y^*) := \{x^* \in X^* \mid (x^*,-y^*) \in N(\operatorname{Gr} F,(\overline{x},\overline{y}))\}.$$

As usual, when $F := f$ is a function, since $\overline{y} \in F(\overline{x})$ means $\overline{y} = f(\overline{x})$, we write $\widehat{D}^*f(\overline{x})$, and similarly for D^*.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be finite at $\overline{x} \in X$ and lower semicontinuous around $\overline{x}$; the Fréchet subdifferential of f at $\overline{x}$ is the set

$$\widehat{\partial} f(\overline{x}) := \{x^* \in X^* \mid (x^*,-1) \in \widehat{N}(\operatorname{epi} f,(\overline{x},f(\overline{x})))\},$$

where $\operatorname{epi} f$ denotes the epigraph of f. The basic (or limiting, or Mordukhovich) subdifferential of f at $\overline{x}$ is given by

$$\partial f(\overline{x}) := \{x^* \in X^* \mid (x^*,-1) \in N(\operatorname{epi} f,(\overline{x},f(\overline{x})))\}.$$

The basic calculus rules for these objects (especially for the subdifferential of the sum) are well-known.

For a cone $K \subset Y$ we denote by

$$K^+ := \{y^* \in Y^* \mid y^*(y) \geq 0,\ \forall y \in K\}$$

its positive dual cone.

Consider the closed subsets $C_1, ..., C_k$ of an Asplund space X. One says that $C_1, ..., C_k$ are allied at $\overline{x} \in C_1 \cap ... \cap C_k$ whenever $(x_{in}) \stackrel{C_i}{\to} \overline{x}$, $x_{in}^* \in \widehat{N}(C_i, x_{in})$, $i = \overline{1,k}$, the relation $(x_{1n}^* + ... + x_{kn}^*) \to 0$ implies $(x_{in}^*) \to 0$ for every $i = \overline{1,k}$. The concept of alliedness was introduced by Penot in [17] in order to get a calculus rule for the Fréchet normal cone to the intersection of sets. More precisely, if the subsets $C_1, ..., C_k$ are allied at $\overline{x} \in C_1 \cap ... \cap C_k$, then there exists $r > 0$ such that, for every $\varepsilon > 0$ and every $x \in [C_1 \cap ... \cap C_k] \cap B_X(\overline{x}, r)$, there exist $x_i \in C_i \cap B_X(x,\varepsilon)$, $i = \overline{1,k}$ such that

$$\widehat{N}(C_1 \cap ... \cap C_k, x) \subset \widehat{N}(C_1, x_1) + ... + \widehat{N}(C_k, x_k) + \varepsilon D_{X^*}.$$

Before providing the first weak openness result with the assumptions given in terms of generalized differentiation objects on dual spaces, we consider the following sets

$$C_1 := \{(x,y,z) \in X \times Y \times Y \mid y \in F(x)\},$$
$$C_2 := \{(x,y,z) \in X \times Y \times Y \mid z \in G(x)\}.$$

Remark that the alliedness of the sets C_1 and C_2 at $(\overline{x},\overline{y},\overline{z}) \in C_1 \cap C_2$ means that for every sequences $(x_n, y_n) \stackrel{\operatorname{Gr} F}{\to} (\overline{x},\overline{y})$, $(u_n, v_n) \stackrel{\operatorname{Gr} G}{\to} (\overline{x},\overline{z})$, and every $x_n^* \in \widehat{D}^*F(x_n,y_n)(y_n^*)$, $u_n^* \in \widehat{D}^*G(u_n,v_n)(v_n^*)$,

$$(x_n^* + u_n^*) \to 0,\ \ (y_n^*) \to 0,\ \ (v_n^*) \to 0 \Rightarrow (x_n^*) \to 0,\ \ (u_n^*) \to 0.$$

Theorem 8 *Let X, Y be Asplund spaces, F, G, $K : X \rightrightarrows Y$ multifunctions with $(\overline{x},\overline{y}) \in \operatorname{Gr} F$, and $(\overline{x},0) \in \operatorname{Gr} G \cap \operatorname{Gr} K$. Assume that the following assumptions are satisfied:*

(i) $\operatorname{Gr} F$, $\operatorname{Gr} G$ and $\operatorname{Gr} K$ are closed around $(\overline{x},\overline{y})$, $(\overline{x},0)$ and $(\overline{x},0)$, respectively;

(ii) the sets C_1 and C_2 are allied at $(\overline{x},\overline{y},0)$;

(iii) there exist $c > 0$, $r > 0$ such that for every $(x_1,y_1) \in \operatorname{Gr} F \cap (B_X(\overline{x},r) \times B_Y(\overline{y},r))$, $(x_2,y_2) \in \operatorname{Gr} G \cap (B_X(\overline{x},r) \times B_Y(0,r))$, $(x_3,y_3) \in \operatorname{Gr} K \cap (B_X(\overline{x},r) \times B_Y(0,r))$, $y^* \in S_{Y^*}$, z_1^*, z_2^*, $z_3^* \in cB_{Y^*}$, $x_1^* \in \widehat{D}^*F(x_1,y_1)(y^* + z_1^*)$, $x_2^* \in \widehat{D}^*G(x_2,y_2)(y^* + z_2^*)$, $x_3^* \in \widehat{D}^*K(x_3,y_3)(y^* + z_3^*)$, we have

$$c\,\|3y^* + z_1^* + z_2^* + z_3^*\| \leq \|x_1^* + x_2^* + x_3^*\|.$$

Then for every $a \in]0,c[$, there exists $\varepsilon > 0$ such that, for every $\rho \in]0,\varepsilon]$

$$B_Y(\overline{y},\rho a) \subset (F+G)(B_X(\overline{x},\rho)) + K(B_X(\overline{x},\rho)),$$

and, consequently, $(F+G,K)$ is weakly open at $(\overline{x},\overline{y})$.

Remark that the previous theorem can be formulated for $(\overline{x},\overline{z}) \in \operatorname{Gr} G$ and $(\overline{x},\overline{w}) \in \operatorname{Gr} K$ instead of $(\overline{x},0) \in \operatorname{Gr} G \cap \operatorname{Gr} K$, but we prefer the present form because of the later use in the paper. Observe that if in Theorem 8 we take $K(x) := 0$ for every $x \in X$, then we obtain Theorem 4.2 from [16], which is an openness result for $F+G$. Furthermore, in case $G(x) := 0$ for every $x \in X$, then we obtain Theorem 5.2 from [12].

Putting together all the facts investigated until now, we are able to formulate necessary optimality conditions for Henig proper nondominated points in VOS setting. The first result, on primal spaces, reads as follows.

Theorem 9 *Let X, Y be Banach spaces, $F, G, K : X \rightrightarrows Y$ be multifunctions such that $\operatorname{Gr} F$ and $\operatorname{Gr} K$ are closed around $(\overline{x},\overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$ and $(\overline{x},0) \in \operatorname{Gr} K$, respectively. Suppose that there exists a neighborhood U of $\overline{x}$ such that:*

(i) there exists $e \in Y \setminus \{0\}$ such that $e \in K(x)$ for all $x \in U$;

(ii) there exists a constant $L > 0$ such that for all $u, v \in U$,

$$F(u) \subset F(v) - L\|u-v\|\, e + K(v);$$

(iii) $K(x)$ is based for all $x \in U$ (the base of $K(x)$ is denoted by $B(x)$);

(iv) there exists a constant $l > 0$ such that for all $u, v \in U$

$$B(u) \subset B(v) + l\|u-v\|\, D_Y.$$

Moreover, suppose that for every $(x,y,t,z) \in \operatorname{Gr} F \times \operatorname{Gr} K$ around $(\overline{x},\overline{y},\overline{x},0) \in \operatorname{Gr} F \times \operatorname{Gr} K$, either K is proto-differentiable at t relative to z, or F is proto-differentiable at x relative to y.

If $(\overline{x},\overline{y}) \in \operatorname{VosHEff}(F,M;K)$, then for every $\varepsilon > 0$, there exist

$$(x_\varepsilon, y_\varepsilon, t_\varepsilon, z_\varepsilon) \in [\operatorname{Gr} F \times \operatorname{Gr} K] \cap [B_X(\overline{x},\varepsilon) \times B_Y(\overline{y},\varepsilon) \times B_X(\overline{x},\varepsilon) \times B_Y(0,\varepsilon)]$$

and $w_\varepsilon \in B_Y(0,\varepsilon) \setminus \{0\}$ such that

$$w_\varepsilon \notin \operatorname{cl}[(D_B F(x_\varepsilon,y_\varepsilon) + D_B(\beta d_M(\cdot)\, e)(x_\varepsilon) \cap B_Y(0,1))(B_X(0,1)) + D_B K(t_\varepsilon,z_\varepsilon)(B_X(0,1)) \cap B_Y(0,1)].$$

In the following we consider as well, besides the sets C_1 and C_2, the set

$$C_3 := \{(x,y,z) \in X \times Y \times Y \mid z \in K(x)\}.$$

We give now our result concerning optimality conditions on dual spaces.

Theorem 10 *Let X be an Asplund space and Y be a finite dimensional space, let $F, K : X \rightrightarrows Y$ be multifunctions such that $\operatorname{Gr} F$ and $\operatorname{Gr} K$ are closed around $(\overline{x},\overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$ and $(\overline{x},0) \in \operatorname{Gr} K$, respectively. Suppose that there exists a neighborhood U of $\overline{x}$ such that:*

(i) there exists $e \in \overline{K} \setminus \{0\}$, where $\overline{K} := \bigcap_{x \in U} K(x)$;

(ii) there exists a constant $L > 0$ such that for all $u, v \in U$,

$$F(u) \subset F(v) - L\|u-v\|\, e + K(v);$$

(iii) $K(x)$ is based for all $x \in U$ (the base of $K(x)$ is denoted by $B(x)$);

(iv) there exists a constant $l > 0$ such that for all $u, v \in U$

$$B(u) \subset B(v) + l\|u-v\|\, D_Y.$$

Moreover, suppose that the sets C_1 and C_3 are allied at $(\overline{x},\overline{y},0)$.

If $(\overline{x},\overline{y}) \in \operatorname{VosHEff}(F,M;K)$, then for all $\beta > L$, there exists $y^* \in \overline{K}^+ \setminus \{0\}$ such that

$$0 \in D^*F(\overline{x},\overline{y})(y^*) + D^*K(\overline{x},0)(y^*) + \beta y^*(e)\, \partial d_M(\overline{x}). \tag{7}$$

In the paper [13], we study some constrained vector optimization problems on an approach in dual spaces, working as well with the generalized differentiation constructions developed by Mordukhovich. More precisely, we study the concept of nondomination, which is also known in the literature as Pareto minimum (or, Pareto efficiency), in constrained multiobjective optimization with respect to variable ordering structures, from the viewpoint of necessary optimality conditions.

We present the vector optimization problem considered, and we define the concept of solution associated to the proposed problem and we give some notations for the generalized differentiation objects used here.

Let m be a nonzero natural number and $F_i : X \rightrightarrows Y_i$ be closed set-valued maps for every $i \in \overline{0,m}$, where X and each Y_i are Asplund spaces. Let $K_0 : X \rightrightarrows Y_0$ be a closed set-valued map such that $K_0(x)$ is a convex, proper and pointed cone in Y_0 for any $x \in X$. Then, for every $x \in X$, the cone $K_0(x)$ introduces an order relation on Y_0 by the equivalence:

$$y_1 \leq_{K_0(x)} y_2 \Leftrightarrow y_2 - y_1 \in K_0(x).$$

For every $i \in \overline{1,m}$ we consider the closed set-valued maps $K_i : X \rightrightarrows Y_i$ such that $K_i(x)$ is a convex and proper cone (not necessarily pointed) in Y_i for any $x \in X$.

We consider the following vector optimization problem

$$\min\nolimits_{K_0} F_0(x), \text{ such that } F_i(x) \cap (-K_i(x)) \neq \emptyset,\ i \in \overline{1,m},\ x \in \Omega, \tag{8}$$

where Ω is a closed and nonempty subset of X.

Take $(\overline{x}, \overline{y}_0, ..., \overline{y}_m) \in \Omega \times \prod_{i=0}^{m} Y_i$ such that $(\overline{x}, \overline{y}_0) \in \operatorname{Gr} F_0$ and, for every $i \in \overline{1,m}$, $\overline{y}_i \in F_i(\overline{x}) \cap -K_i(\overline{x})$. Let $S \subset X$ denote the feasible set of (8), that is,

$$S := \left\{x \in \Omega \colon 0 \in (F_i + K_i)(x),\ \forall i \in \overline{1,m}\right\}.$$

The point $(\overline{x}, \overline{y}_0)$ is a local solution of (8) if there exists $U \in \mathcal{V}(\overline{x})$ such that for every $x \in S \cap U$ one has

$$(F_0(x) - \overline{y}_0) \cap (-K_0(x)) \subset \{0\}$$

(for more details on this notion, see [11]).

We reconsider the main ideas developed in [20] for the case of a functional constrained vector optimization problem with fixed order structure, to the case of variable order structures. In this demarche, we have to solve several technical questions that arise in the latter, more general case. In particular, the investigation we propose has to take into account the presence of the set-valued map that defines the order with respect to which the efficiency concept we work with is designed, namely K_0, and also the set-valued maps that appear in the definition of the constraints, i.e., K_i with $i \in \overline{1,m}$.

The main tool used in order to achieve our main goal is the following version of an extended extremal principal (for more details see [20, Lemma 2.1]).

Lemma 11 *Let Y be an Asplund space and $A_1, A_2, ..., A_n \subset Y$ be closed sets such that $\bigcap_{i=1}^{n} A_i = \emptyset$. Let $a_i \in A_i$ (for $i \in \overline{1,n}$) and $\varepsilon > 0$ be such that*

$$\sum_{i=1}^{n-1} \|a_i - a_n\| < \gamma(A_1, A_2, ..., A_n) + \varepsilon,$$

where

$$\gamma(A_1, A_2, ..., A_n) := \inf\left\{\sum_{i=1}^{n-1} \|a_i - a_n\| : (a_1, a_2, ..., a_n) \in A_1 \times A_2 \times ... \times A_n\right\}.$$

Then for every $\lambda > 0$, there exist $\widetilde{a}_i \in A_i$ and $a_i^* \in Y^*$ such that

$$\sum_{i=1}^{n} \|a_i - \widetilde{a}_i\| < \lambda,\ a_i^* \in \widehat{N}(A_i, \widetilde{a}_i) + \frac{\varepsilon}{\lambda} D_{Y^*},$$

$$\sum_{i=1}^{n} \|a_i^*\| = 1 \text{ and } \sum_{i=1}^{n} a_i^* = 0.$$

In the above notations, applying Lemma 11 for the constants $\varepsilon = \frac{1}{k^2}$ and $\lambda = \frac{1}{k}$, with $k \in \mathbb{N}^*$, the points $a_i := (\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m)$, for every $i \in \overline{0, 2m+1}$, $a_{2m+2} := (\overline{x}, \overline{y}_0 - s_k c_0, \overline{y}_1, ..., \overline{y}_m)$ and the sets

$$A_i := \left\{ (x, y_0, y_1, ..., y_m) \in X \times \prod_{i=0}^{m} Y_i \colon y_i \in F_i(x) \right\}, \ \forall i \in \overline{0,m},$$
$$A_{m+i} := \left\{ (x, y_0, y_1, ..., y_m) \in X \times \prod_{i=0}^{m} Y_i \colon y_i \in -K_i(x) \right\}, \ \forall i \in \overline{1,m},$$
$$A_{2m+1} := \left\{ (x, y_0, y_1, ..., y_m) \in X \times \prod_{i=0}^{m} Y_i \colon x \in \Omega \cap D(\overline{x}, \delta) \right\},$$
$$A_{2m+2} := \left\{ (x, y_0, y_1, ..., y_m) \in X \times \prod_{i=0}^{m} Y_i \colon y_0 \in \overline{y}_0 - s_k c_0 - K_0(x) \right\}.$$

we are able to obtain, by using the Fréchet coderivatives of the set-valued maps F_i and K_i for every $i \in \overline{0,m}$ and the Fréchet normal cone to Ω, the following result concerning some necessary optimality conditions for a solution of problem (8).

Theorem 12 *Let $(\overline{x}, \overline{y}_0) \in \operatorname{Gr} F_0 \cap (\Omega \times Y_0)$ be a local solution for problem (8) and the points $\overline{y}_i \in F_i(\overline{x}) \cap -K_i(\overline{x})$ for every $i \in \overline{1,m}$. Suppose that there exists $W \in \mathcal{V}(\overline{x})$ such that $\bigcap_{x \in \Omega \cap W} K_0(x) \neq \{0\}$. Then one of the following assertions holds:*

(i) for every $\varepsilon > 0$, $i \in \overline{0,m}$, and $j \in \overline{1,m}$, there exist $x_i, \widetilde{x}_0, \widetilde{x}_j \in \overline{x} + \varepsilon D_X$, $y_i \in (\overline{y}_i + \varepsilon D_{Y_i}) \cap F_i(x_i)$, $\widetilde{y}_0 \in (\varepsilon D_{Y_0}) \cap K_0(\widetilde{x}_0)$, $\widetilde{y}_j \in \left(-\overline{y}_j + \varepsilon D_{Y_j}\right) \cap K_j(\widetilde{x}_j)$, $\omega \in \Omega \cap (\overline{x} + \varepsilon D_X)$, $y_i^* \in Y_i^*$ such that $\sum_{i=0}^{m} \|y_i^*\| = 1$ and

$$0 \in \sum_{i=0}^{m} \left(\widehat{D}^* F_i(x_i, y_i) \left(y_i^* + \varepsilon D_{Y_i^*}\right) \cap M B_{X^*} \right) + \sum_{i=0}^{m} \left(\widehat{D}^* K_i(\widetilde{x}_i, \widetilde{y}_i) \left(y_i^* + \varepsilon D_{Y_i^*}\right) \cap M B_{X^*} \right)$$
$$+ \widehat{N}(\Omega, \omega) \cap M B_{X^*} + \varepsilon D_{X^*},$$

where $M > 0$ is a constant independent of ε;

(ii) for every $\varepsilon > 0$, $i \in \overline{0,m}$, and $j \in \overline{1,m}$, there exist $x_i, \widetilde{x}_0, \widetilde{x}_j \in \overline{x} + \varepsilon D_X$, $y_i \in (\overline{y}_i + \varepsilon D_{Y_i}) \cap F_i(x_i)$, $\widetilde{y}_0 \in (\varepsilon D_{Y_0}) \cap K_0(\widetilde{x}_0)$, $\widetilde{y}_j \in \left(-\overline{y}_j + \varepsilon D_{Y_j}\right) \cap K_j(\widetilde{x}_j)$, $\omega \in \Omega \cap (\overline{x} + \varepsilon D_X)$, $x_i^* \in \widehat{D}^* F_i(x_i, y_i)(\varepsilon D_{Y_i^*})$, $\widetilde{x}_i^* \in \widehat{D}^* K_i(\widetilde{x}_i, \widetilde{y}_i) \left(\varepsilon D_{Y_i^*}\right)$, $\omega^* \in \widehat{N}(\Omega, \omega) + \varepsilon D_{X^*}$ such that

$$\|\omega^*\| + \sum_{i=0}^{m} (\|x_i^*\| + \|\widetilde{x}_i^*\|) = 1 \text{ and } \omega^* + \sum_{i=0}^{m} (x_i^* + \widetilde{x}_i^*) = 0.$$

Further, we proved that the conclusion (ii) of Theorem 12 does not hold, when we impose some alliedness hypothesis and $\Omega = X$.

Corollary 13 *Let $(\overline{x}, \overline{y}_0) \in \operatorname{Gr} F_0$ be a local solution for problem (8) and the points $\overline{y}_i \in F_i(\overline{x}) \cap -K_i(\overline{x})$ for every $i \in \overline{1,m}$. Suppose that there exists W a neighborhood of $\overline{x}$ such that $\bigcap_{x \in W} K_0(x) \neq \{0\}$. If the closed sets $C_0, C_1, ..., C_{2m+1}$ are allied at $(\overline{x}, \overline{y}_0, ..., \overline{y}_m, 0, -\overline{y}_1, -\overline{y}_2, ..., -\overline{y}_m) \in \bigcap_{i=0}^{2m+1} C_i$, where for every $i \in \overline{0,m}$,*

$$C_i := \left\{ (x, y_0, y_1, ..., y_{2m+1}) \in X \times Y^{2m+2} \colon y_i \in F_i(x) \right\}, \tag{9}$$
$$C_{m+i+1} := \left\{ (x, y_0, y_1, ..., y_{2m+1}) \in X \times Y^{2m+2} \colon y_{m+i+1} \in K_i(x) \right\},$$

then for every $\varepsilon > 0$, $i \in \overline{0,m}$, and $j \in \overline{1,m}$, there exist $x_i, \widetilde{x}_0, \widetilde{x}_j \in \overline{x} + \varepsilon D_X$, $y_i \in (\overline{y}_i + \varepsilon D_{Y_i}) \cap F_i(x_i)$, $\widetilde{y}_0 \in (\varepsilon D_{Y_0}) \cap K_0(\widetilde{x}_0)$, $\widetilde{y}_j \in \left(-\overline{y}_j + \varepsilon D_{Y_j}\right) \cap K_j(\widetilde{x}_j)$, $\omega \in \Omega \cap (\overline{x} + \varepsilon D_X)$, $y_i^* \in Y_i^*$ such that $\sum_{i=0}^{m} \|y_i^*\| = 1$ and

$$0 \in \sum_{i=0}^{m} \left(\widehat{D}^* F_i(x_i, y_i) \left(y_i^* + \varepsilon D_{Y_i^*}\right) \cap M B_{X^*} \right) + \sum_{i=0}^{m} \left(\widehat{D}^* K_i(\widetilde{x}_i, \widetilde{y}_i) \left(y_i^* + \varepsilon D_{Y_i^*}\right) \cap M B_{X^*} \right) + \varepsilon D_{X^*}, \tag{10}$$

where $M > 0$ is a constant independent of ε.

In the following, we tackle again the general case and we present some necessary optimality conditions for a local solution of problem (8) using Mordukhovich coderivatives and the basic normal cone. Mention that for this passing from the approximate optimality conditions presented in Theorem 12, to exact optimality conditions we have to impose some specific conditions of sequential normal compactness (for more details, see [15, pages 27, 76, 266]).

Theorem 14 *Let $(\overline{x}, \overline{y}_0) \in \operatorname{Gr} F_0 \cap (\Omega \times Y_0)$ be a local solution for problem (8) and the points $\overline{y}_i \in F_i(\overline{x}) \cap -K_i(\overline{x})$ for every $i \in \overline{1,m}$. Suppose that there exists $W \in \mathcal{V}(\overline{x})$ such that $\overline{K}_i := \bigcap_{x \in \Omega \cap W} K_i(x) \neq \{0\}$ for every $i \in \overline{0,m}$. If for every $i \in \overline{0,m}$, $j \in \overline{1,m}$ the sets $\overline{K}_i$ are sequentially normally compact in 0 and the set-valued maps F_i, K_0 and K_j are partially sequentially normally compact in $(\overline{x}, \overline{y}_i) \in \operatorname{Gr} F_i$, $(\overline{x}, 0) \in \operatorname{Gr} K_0$, respectively $(\overline{x}, -\overline{y}_j) \in \operatorname{Gr} K_j$, then one of the following assertions holds:*

(i) for every $i \in \overline{0,m}$, there exist $y_i^* \in \overline{K}_i^+$ such that $\sum_{i=0}^{m} \|y_i^*\| = 1$ and

$$0 \in \sum_{i=1}^{m} \left[D^* F_i(\overline{x}, \overline{y}_i)(y_i^*) + D^* K_i(\overline{x}, -\overline{y}_i)(y_i^*)\right] + D^* F_0(\overline{x}, \overline{y}_0)(y_0^*) + D^* K_0(\overline{x}, 0)(y_0^*) + N(\Omega, \overline{x}),$$

where $\overline{K}_i^+$ denotes the positive dual cone of K_i;

(ii) for every $i \in \overline{0,m}$, and $j \in \overline{1,m}$, there exist $x_i^* \in D^* F_i(\overline{x}, \overline{y}_i)(0)$, $\widetilde{x}_0^* \in D^* K_0(\overline{x}, 0)(0)$, $\widetilde{x}_j^* \in D^* K_j(\overline{x}, -\overline{y}_j)(0)$, $\omega^* \in N(\Omega, \overline{x})$ such that

$$\omega^* + \sum_{i=0}^{m} (x_i^* + \widetilde{x}_i^*) = 0 \text{ and } \|\omega^*\| + \sum_{i=0}^{m} (\|x_i^*\| + \|\widetilde{x}_i^*\|) = 1. \tag{11}$$

In order to illustrate Theorem 14, we consider the following examples.
Similar to Corollary 13, under some alliedness conditions and for $\Omega = X$, one gets the following result.

Corollary 15 *Let $(\overline{x}, \overline{y}_0) \in \operatorname{Gr} F_0$ be a local solution for problem (8) and the points $\overline{y}_i \in F_i(\overline{x}) \cap -K_i(\overline{x})$ for every $i \in \overline{1,m}$. Suppose that there exists W a neighborhood of $\overline{x}$ such that $\overline{K}_i := \bigcap_{x \in W} K_i(x) \neq \{0\}$ for every $i \in \overline{0,m}$. If for every $i \in \overline{0,m}$ the sets $\overline{K}_i$ are sequentially normally compact in 0 and the closed sets $C_0, C_1, ..., C_{2m+1}$ are allied at $(\overline{x}, \overline{y}_0, ..., \overline{y}_m, 0, -\overline{y}_1, -\overline{y}_2, ..., -\overline{y}_m) \in \bigcap_{i=0}^{2m+1} C_i$, where the sets $C_0, C_1, ..., C_{2m+1}$ are given in (9), then for every $i \in \overline{0,m}$, there exist $y_i^* \in \overline{K}_i^+$ such that $\sum_{i=0}^{m} \|y_i^*\| = 1$ and*

$$0 \in \sum_{i=1}^{m} \left[D^* F_i(\overline{x}, \overline{y}_i)(y_i^*) + D^* K_i(\overline{x}, -\overline{y}_i)(y_i^*)\right] + D^* F_0(\overline{x}, \overline{y}_0)(y_0^*) + D^* K_0(\overline{x}, 0)(y_0^*).$$

By simply taking the particular cases our approach cover, we are able to reobtain several results from the literature, in their original forms or in some variations.

In the paper [2] we select two tools of investigation of the classical metric regularity of set-valued mappings, namely the Ioffe criterion and the Ekeland Variational Principle, which we adapt to the study of the directional setting. In this way, we obtain in a unitary manner new and generalized results concerning sufficient conditions for directional metric regularity of a mapping, with applications to the stability of this property at composition and sum of set-valued maps. In this process, we introduce as well new directional tangent cones and the associated generalized differentiation objects and concepts on primal spaces. Moreover, we underline several links between our main assertions by providing alternative proofs for several results.

Let X be a normed vector space, $\emptyset \neq \Omega \subset X$ and $\emptyset \neq L \subset S_X$. Then the function

$$X \ni x \longmapsto T_L(x, \Omega) := \inf \{t \geq 0 \mid \exists \ell \in L : x + t\ell \in \Omega\} \tag{12}$$

is called the directional minimal time function with respect to L. Many properties of this function were systematically analyzed in [9]. Remark that

$$T_L(x,\Omega) < +\infty \text{ if and only if } x \in \Omega - \operatorname{cone} L$$

and

$$d(x,\Omega) \leq T_L(x,\Omega) \text{ for all } x \in X.$$

Moreover, if $L = S_X$, then $T_L(\cdot,\Omega) = d(\cdot,\Omega)$. We shall sometimes use the notation T_L instead of $T_L(\cdot,\Omega)$, when no danger of confusion arises. Moreover, if $\Omega = \{u\}$ for a point $u \in X$, we denote in what follows $T_L(\cdot,\{u\})$ by $T_L(\cdot,u)$. Clearly, for each x, $u \in X$, if $T_L(x,u) < +\infty$ (which is equivalent to $u - x \in \operatorname{cone} L$), then

$$T_{-L}(u,x) = T_L(x,u) = \|u - x\|.$$

Definition 16 *Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ with $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ and sets $L \subset S_X$ and $M \subset S_Y$ be nonempty.*

(i) One says that F is directionally metrically regular around $(\overline{x},\overline{y})$ with respect to L and M with a constant $c > 0$ *if there are $\varepsilon > 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that, for every $(x,y) \in U \times V$ such that $T_M(y,F(x)) < \varepsilon$,*

$$T_L(x,F^{-1}(y)) \leq c \cdot T_M(y,F(x)). \tag{13}$$

The modulus of directional regularity of F around $(\overline{x},\overline{y})$ with respect to L and M, *denoted by* $\operatorname{dirreg}_{L\times M} F(\overline{x},\overline{y})$, *is the infimum of $c > 0$ such that F is directionally metrically regular around $(\overline{x},\overline{y})$ with respect to L and M with the constant c.*

(ii) One says that F is directionally linearly open around $(\overline{x},\overline{y})$ with respect to L and M with a constant $c > 0$ *if there are $\varepsilon > 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that, for every $r \in (0,\varepsilon)$ and every $(x,y) \in (U \times V) \cap \operatorname{Gr} F$,*

$$B(y,cr) \cap (y - \operatorname{cone} M) \subset F(B(x,r) \cap (x + \operatorname{cone} L)). \tag{14}$$

The modulus of directional openness of F around $(\overline{x},\overline{y})$ with respect to L and M, *denoted by* $\operatorname{dirsur}_{L\times M} F(\overline{x},\overline{y})$, *is the supremum of $c > 0$ such that F is directionally linearly open around $(\overline{x},\overline{y})$ with respect to L and M with the constant c.*

(iii) One says that F has the directional Aubin property around $(\overline{x},\overline{y})$ with respect to L and M with a constant $c > 0$ *if there are neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that, for every $x, u \in U$,*

$$e_M(F(x) \cap V, F(u)) \leq c \cdot T_L(u,x). \tag{15}$$

The modulus of the directional Aubin property of F around $(\overline{x},\overline{y})$ with respect to L and M, *denoted by* $\operatorname{dirlip}_{L\times M} F(\overline{x},\overline{y})$, *is the infimum of $c > 0$ such that F has the directional Aubin property around $(\overline{x},\overline{y})$ with respect to L and M with the constant c.*

Of course, when $L := S_X$ and $M := S_Y$, the previous concepts reduce to the usual metric regularity, linear openness, and Aubin property around the reference point (see, e.g., [4] for more details).

The next result contains the announced link between the notions given before (see [10, Proposition 2.3]). The convention $1/0 = +\infty$ applies here.

Proposition 17 *Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ with $(\overline{x},\overline{y}) \in \operatorname{Gr} F$and sets $L \subset S_X$ and $M \subset S_Y$ be nonempty. Then*

$$\operatorname{dirreg}_{L\times M} F(\overline{x},\overline{y}) = (\operatorname{dirsur}_{L\times M} F(\overline{x},\overline{y}))^{-1} = \operatorname{dirlip}_{M\times L} F^{-1}(\overline{y},\overline{x}).$$

Moreover, a direct comparison of these concept with other directional concepts of regularity is given.

The first Ioffe type criterion is given next for single-valued maps

Proposition 18 *Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ be Banach spaces. Consider a nonempty closed subset L of S_X such that* $\operatorname{cone} L$ *is convex, a nonempty closed subset M of S_Y, a point $\overline{x} \in X$, and a mapping $g : X \to Y$ such that there is a neighborhood U of $\bar{x}$ such that the set $D := U \cap \operatorname{Dom} g$ is closed and g is continuous on D. Then* $\operatorname{dirsur}_{L\times M} g(\overline{x})$ *equals to the supremum of $c > 0$ for which there is $r > 0$ such that for all $(x,y) \in (B[\overline{x},r] \cap \operatorname{Dom} g) \times B[g(\overline{x}),r]$, with $0 < T_M(y,g(x)) < +\infty$, there is a point $x' \in \operatorname{Dom} g$ satisfying*

$$cT_L(x,x') < T_M(y,g(x)) - T_M(y,g(x')).$$

For set-valued maps we derive the following assertion.

Proposition 19 *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces. Consider nonempty closed subsets L of S_X and M of S_Y such that* $\operatorname{cone} L$ *and* $\operatorname{cone} M$ *are convex, a point $(\overline{x}, \overline{y}) \in X \times Y$, and a set-valued mapping $F : X \rightrightarrows Y$ the graph of which is locally closed near $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Then $\operatorname{dirsur}_{L\times M} F(\overline{x}, \overline{y})$ equals to the supremum of all $c > 0$ for which there are $r > 0$ and $\alpha \in (0, 1/c)$ such that for any $(x, v) \in (B[\overline{x}, r] \times B[\overline{y}, r]) \cap \operatorname{Gr} F$ and any $y \in B[\overline{y}, r]$, with $0 < T_M(y, v) < +\infty$, there is a pair $(x', v') \in \operatorname{Gr} F$ such that*

$$c \max\{T_L(x, x'), \alpha\|v - v'\|\} < T_M(y, v) - T_M(y, v'). \tag{16}$$

In what follows, we speak about the local stability at composition of a pair multifunctions, which essentially says that a point from the graph of the composed multifunction, close to the reference one, can be written by the use of points from the graphs of the involved set-valued maps, which are also close to the corresponding reference ones. Given metric spaces (X, ϱ), (Y, ϱ), and (Z, ϱ), a composition of set-valued mappings $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ is the mapping $G \circ F : X \rightrightarrows Z$ defined by

$$(G \circ F)(x) := \bigcup_{y \in F(x)} G(y), \quad x \in X;$$

and a product of set-valued mappings $F_1 : X \rightrightarrows Y$ and $F_2 : X \rightrightarrows Z$ is the mapping $(F_1, F_2) : X \rightrightarrows Y \times Z$ defined by

$$(F_1, F_2)(x) := F_1(x) \times F_2(x), \quad x \in X.$$

Definition 20 *Let (X, ϱ), (Y, ϱ), and (Z, ϱ) be metric spaces and $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Z$ be fixed. Consider set-valued mappings $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ such that $\overline{y} \in F(\overline{x})$ and $\overline{z} \in G(\overline{y})$. We say that the pair F, G is* composition-stable around $(\overline{x}, \overline{y}, \overline{z})$ *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $x \in B(\overline{x}, \delta)$ and every $z \in (G \circ F)(x) \cap B(\overline{z}, \delta)$, there exists $y \in F(x) \cap B(\overline{y}, \varepsilon)$ such that $z \in G(y)$.*

We present next one of the main results, which asserts the stability of directional regularity under composition.

Theorem 21 *Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$, $(Z, \|\cdot\|)$, and $(W, \|\cdot\|)$ be Banach spaces and $(\overline{x}, \overline{y}, \overline{z}, \overline{w}) \in X \times Y \times Z \times W$ be fixed. Consider nonempty closed subsets L of S_X, M of S_Y, N of S_Z, and P of S_W such that* $\operatorname{cone} L$, $\operatorname{cone} M$, $\operatorname{cone} N$, *and* $\operatorname{cone} P$ *are convex, set-valued mappings $F_1 : X \rightrightarrows Y$, $F_2 : X \rightrightarrows Z$, and $G : Y \times Z \rightrightarrows W$ such that F_1 has a locally closed graph near $(\overline{x}, \overline{y}) \in \operatorname{Gr} F_1$, F_2 has a locally closed graph near $(\overline{x}, \overline{z}) \in \operatorname{Gr} F_2$, and G has a locally closed graph near $(\overline{y}, \overline{z}, \overline{w}) \in \operatorname{Gr} G$. Define the mapping $\mathcal{E}_{G,(F_1,F_2)} : X \times Y \times Z \rightrightarrows W$ by*

$$\mathcal{E}_{G,(F_1,F_2)}(x, y, z) := \begin{cases} G(y, z), & \textit{if } (y, z) \in (F_1, F_2)(x), \\ \emptyset, & \textit{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \operatorname{dirsur}_{L\times M\times N\times P} \mathcal{E}_{G,(F_1,F_2)}(\overline{x}, \overline{y}, \overline{z}, \overline{w}) \geq{}& \operatorname{dirsur}_{L\times M} F_1(\overline{x}, \overline{y}) \cdot \widehat{\operatorname{dirsur}}^{\,y}_{-M\times P} G(\overline{y}, \overline{z}, \overline{w}) \\ &- \operatorname{dirlip}_{-L\times N} F_2(\overline{x}, \overline{z}) \cdot \widehat{\operatorname{dirlip}}^{\,z}_{-N\times P} G(\overline{y}, \overline{z}, \overline{w}). \end{aligned} \tag{17}$$

If, in addition, the pair $(F_1, F_2), G$ is composition-stable around $(\overline{x}, (\overline{y}, \overline{z}), \overline{w})$, then

$$\begin{aligned} \operatorname{dirsur}_{L\times P} (G \circ (F_1, F_2))(\overline{x}, \overline{w}) \geq{}& \operatorname{dirsur}_{L\times M} F_1(\overline{x}, \overline{y}) \cdot \widehat{\operatorname{dirsur}}^{\,y}_{-M\times P} G(\overline{y}, \overline{z}, \overline{w}) \\ &- \operatorname{dirlip}_{-L\times N} F_2(\overline{x}, \overline{z}) \cdot \widehat{\operatorname{dirlip}}^{\,z}_{-N\times P} G(\overline{y}, \overline{z}, \overline{w}). \end{aligned} \tag{18}$$

Moreover, we employ a directional version of the Bouligand (graphical) derivative to derive sufficient conditions for the directional regularity.

Definition 22 *Let Ω be a nonempty subset of a normed space $(X, \|\cdot\|)$, $M \subset S_X$ be a nonempty set, and $\overline{x} \in \Omega$. The* Bouligand-Severi tangent cone to Ω at $\overline{x}$ with respect to M *is the set*

$$T(\Omega, \overline{x}, M) = \left\{ u \in X \mid \liminf_{t \downarrow 0} \; t^{-1} T_M(\overline{x} + tu, \Omega) = 0 \right\}. \tag{19}$$

Similar to the classical case, one can introduce the directional Ursescu cone, as follows.

Definition 23 *Let Ω be a nonempty subset of a normed space $(X, \|\cdot\|)$, $M \subset S_X$ be a nonempty set, and $\overline{x} \in \Omega$. The* adjacent cone to Ω at $\overline{x}$ with respect to M *is the set*

$$T^\flat(\Omega, \overline{x}, M) = \left\{ u \in X \mid \lim_{t \downarrow 0} \; t^{-1} T_M(\overline{x} + tu, \Omega) = 0 \right\}, \tag{20}$$

that is,

$$T^\flat(\Omega, \overline{x}, M) = \{u \in X \mid \forall (t_n) \downarrow 0, \; \exists (u_n) \subset u + \operatorname{cone} M, \; u_n \to u, \; \forall n \in \mathbb{N}, \; \overline{x} + t_n u_n \in \Omega\}.$$

It is clear that, in general,

$$T^\flat(\Omega, \overline{x}, M) \subset T(\Omega, \overline{x}, M).$$

Definition 24 *Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ with $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, $L \subset S_X$ and $M \subset S_Y$ be nonempty sets.*

(i) The Bouligand derivative of F at $(\overline{x}, \overline{y})$ with respect to L and M *is the set-valued mapping $D_{L,M}F(\overline{x}, \overline{y})$ from X into Y defined by*

$$\begin{aligned} D_{L,M}F(\overline{x}, \overline{y})(u) =& \{v \in Y \mid \exists (t_n) \downarrow 0, \; \exists (u_n) \subset u + \operatorname{cone} L, \; u_n \to u, \; \exists (v_n) \subset v + \operatorname{cone} M, \; (v_n) \to v, \\ & \forall n \in \mathbb{N}, \; \overline{y} + t_n v_n \in F(\overline{x} + t_n u_n)\}. \end{aligned}$$

(ii) The adjacent derivative of F at $(\overline{x}, \overline{y})$ with respect to L and M *is the set-valued mapping denoted $D^\flat_{L,M}F(\overline{x}, \overline{y})$ from X into Y defined similarly.*

(iii) One says that F is directionally proto-differentiable with respect to $L \times M$ at $\overline{x}$ relative to $\overline{y}$ if $D_{L,M}F(\overline{x}, \overline{y}) = D^\flat_{L,M}F(\overline{x}, \overline{y})$.

Observe that if $\operatorname{cone} \widetilde{L} = \operatorname{cone} L \times \operatorname{cone} M$, with an appropriate choice of $\widetilde{L} \subset S_{X \times Y}$, then

$$\begin{aligned} &\operatorname{Gr} D_{L,M}F(\overline{x}, \overline{y}) = T(\operatorname{Gr} F, (\overline{x}, \overline{y}), \widetilde{L}) \text{ and} \\ &\operatorname{Gr} D^\flat_{L,M}F(\overline{x}, \overline{y}) = T^\flat(\operatorname{Gr} F, (\overline{x}, \overline{y}), \widetilde{L}). \end{aligned}$$

One has the following statement.

Theorem 25 *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces. Consider nonempty closed subsets L of S_X and M of S_Y such that $\operatorname{cone} L$ and $\operatorname{cone} M$ are convex and a mapping $F : X \rightrightarrows Y$ the graph of which is locally closed near $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Assume that there are positive constants β, ϱ, and r such that for every $(x, v) \in (B[\overline{x}, r] \times B[\overline{y}, r]) \cap \operatorname{Gr} F$ we have*

$$D_{L,M}F(x, v)(\mathbb{B}_X \cap \operatorname{cone} L) + B[0, \beta] \cap (-\operatorname{cone} M) \supset -(\beta + \varrho)M.$$

Then $\operatorname{dirsur}_{L \times M} F(\overline{x}, \overline{y}) \geq \varrho$.

Finally, we formulate results that use Theorem 25 in order to give primal sufficient conditions for the directional metric regularity of compositions and sums. Note that the next theorem is new even for the non-directional case. For the next results, in the notation of Definition 5, we denote by $D_{L,M}F(x, y) \cap \mathbb{B}_Y \cap \operatorname{cone} M$ the multifunction $H : X \rightrightarrows Y$ given by

$$H(a) = D_{L,M}F_1(x, y)(a) \cap \mathbb{B}_Y \cap \operatorname{cone} M.$$

Theorem 26 *Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$, $(Z, \|\cdot\|)$, and $(W, \|\cdot\|)$ be Banach spaces and $(\overline{x}, \overline{y}, \overline{z}, \overline{w}) \in X \times Y \times Z \times W$ be fixed. Consider the nonempty closed subsets L of S_X, M of S_Y, N of S_Z, and P of S_W such that* $\operatorname{cone} L$, $\operatorname{cone} M$, $\operatorname{cone} N$, *and* $\operatorname{cone} P$ *are convex, the set-valued mappings $F_1 : X \rightrightarrows Y$, $F_2 : X \rightrightarrows Z$, and $G : Y \times Z \rightrightarrows W$ such that F_1 has a locally closed graph near $(\overline{x}, \overline{y}) \in \operatorname{Gr} F_1$, F_2 has a locally closed graph near $(\overline{x}, \overline{z}) \in \operatorname{Gr} F_2$, and G has a locally closed graph near $(\overline{y}, \overline{z}, \overline{w}) \in \operatorname{Gr} G$. Suppose, moreover, that there exist the positive constants β, ϱ, and r such that for every $(x, y, z, w) \in (B[\overline{x}, r] \times B[\overline{y}, r] \times B[\overline{z}, r] \times B[\overline{w}, r]) \cap \operatorname{Gr} \mathcal{E}_{G,(F_1,F_2)}$ we have:*

(i) the next relation holds

$$D_{M,N,P}G(y,z,w)(D_{L,M}F_1(x,y)\cap \mathbb{B}_Y\cap \operatorname{cone} M, D_{L,N}F_2(x,z)\cap \mathbb{B}_Z\cap \operatorname{cone} N)(\mathbb{B}_X\cap \operatorname{cone} L)$$
$$+B[0,\beta]\cap(-\operatorname{cone} P)\supset -(\beta+\varrho)P;$$

(ii) either F_1 is directionally proto-differentiable with respect to $L\times M$ at x relative to y or F_2 is directionally proto-differentiable with respect to $L\times N$ at x relative to z;

(iii) either F_1 has the directional Aubin property with respect to S_X and M around (x,y) or F_2 has the directional Aubin property with respect to S_X and N around (x,z);

(iv) G is directionally proto-differentiable with respect to $M\times N\times P$ at (y,z) relative to w;

(v) G has the directional Aubin property with respect to $S_Y\times S_Z$ and P around (y,z,w);

(vi) the pair $(F_1,F_2),G$ is composition-stable around $(x,(y,z),w)$.

Then $\operatorname{dirsur}_{L\times P}[G\circ(F_1,F_2)](\overline{x},\overline{w})\geq\varrho$.

The paper ends with a full description of the implications between the main analytical tools we use.

The paper [8] continues the investigation from [7] and we present a barrier method for vector optimization problems with inequality constraints. To this aim, we firstly investigate some constraint qualification conditions and we compare them to the corresponding ones in literature. Then, we define a barrier function and observe that its basic properties do work for fairly general situations, while for meaningful convergence results of the associated barrier method we should restrict ourselves to convex case and finite dimensional setting.

Let X,Y,Z be Banach spaces and let us consider on Y and Z some partial order relations given by the closed convex and pointed cones $K\subset Y$ and $Q\subset Z$, respectively. More precisely, on Y we have the relation $\leq_K$ given by the equivalence $y_1\leq_K y_2$ iff $y_2-y_1\in K$ and, similarly, the relation $\leq_Q$ on Z. Moreover, we suppose that $\operatorname{int} K\neq\emptyset$ and $\operatorname{int} Q\neq\emptyset$.

Take $f:X\to Y$ and $g:X\to Z$ as vectorial continuous single-valued mappings and consider the following vector optimization problem:

$$(P)\ \min f(x) \text{ s.t. } g(x)\in -Q.$$

Here, in view of the assumptions on the cone K, the optimality is understood in the weak Pareto sense: $\overline{x}$ is a weak solution of the problem (P) if for any x satisfying the constraint (i.e., $g(x)\in -Q$), $f(x)-f(\overline{x})\notin -\operatorname{int} K$. Observe that the restriction $g(x)\in -Q$ is a generalized inequality constraint, since in the situation $Z:=\mathbb{R}^p$, $Q:=\mathbb{R}^p_+=[0,\infty)^p$ this reduces to the system of inequalities $g_i(x)\leq 0$ for all $i\in\overline{1,p}$ where g_i are the coordinates functions of g.

We aim at introducing a barrier function associated to (P) and for that, first of all, we have to consider the set M of feasible points, that is, $M:=\{x\in X\mid g(x)\in -Q\}$, the set $\operatorname{int} M$ and the set $\operatorname{strict} M:=\{x\in X\mid g(x)\in -\operatorname{int} Q\}$. In view of the continuity of g and the closedness of Q, the set M is closed, while the sets $\operatorname{int} M$, $\operatorname{strict} M$ are obviously open. Clearly, $\operatorname{strict} M\subset\operatorname{int} M$. In some situations, for a proper definition of the barrier function we envisage, one needs to have the equality $\operatorname{strict} M\subset\operatorname{int} M$ which, in general, does not hold. The fact that $\operatorname{strict} M\neq\emptyset$ is nothing else but the well-known Slater condition: there exists $x\in X$ with $g(x)\in -\operatorname{int} Q$.

In order to formulate the result that ensures the equality between $\operatorname{int} M$ and $\operatorname{strict} M$, we need the following definition: one says that g is open at a point $x\in X$ if for every $\varepsilon>0$ there exists $\delta>0$ such that $B(g(x),\delta)\subset g(B(x,\varepsilon))$.

Lemma 27 *Suppose that g is open at every point x for which $g(x)\in -\operatorname{bd} Q$. Then* $\operatorname{strict} M=\operatorname{int} M$.

Another preparatory discussion concern the main tool we use to scalarize both the objective function and the constraint system of problem (P). More precisely, we use the Gerstewitz functional in the special case when the ordering cone has nonempty interior.

Theorem 28 *Let $K\subset Y$ be a closed convex cone with nonempty interior. Then for every $e\in\operatorname{int} K$ the functional $s_{K,e}:Y\to\mathbb{R}$ given by*

$$s_{K,e}(y)=\inf\{\lambda\in\mathbb{R}\mid \lambda e\in y+K\} \tag{21}$$

is convex continuous and for every $\lambda\in\mathbb{R}$,

$$\{y\in Y\mid s_{K,e}(y)<\lambda\}=\lambda e-\operatorname{int} K,\ \textit{and}\ \{y\in Y\mid s_{K,e}(y)=\lambda\}=\lambda e-\operatorname{bd} K. \tag{22}$$

*Moreover, $s_{K,e}$ is sublinear, $K-$monotone (that is, for all $y_1, y_2 \in Y$, $y_1 \leq_K y_2$ implies $f(y_1) \leq f(y_2)$), strictly-*int K*-monotone (that is, for all $y_1, y_2 \in Y$, $y_2 - y_1 \in \operatorname{int} K$ implies $f(y_1) < f(y_2)$) and for every $u \in Y$, the Fenchel (convex) subdifferential $\partial s_{K,e}(u)$ is nonempty and*

$$\partial s_{K,e}(u) = \{v^* \in K^* \mid v^*(e) = 1, v^*(u) = s_{K,e}(u)\}.$$

In addition, $s_{K,e}$ is $d(e, \operatorname{bd}(K))^{-1}$–Lipschitzian.

The next lemma links the minima of the composite function $s_{K,e} \circ f$ with the minimizers of f.

Lemma 29 *If $\overline{x} \in M$ is a minimum on M of the scalar function $s_{K,e} \circ f$, then it is a weak solution of the problem (P) as well.*

Another useful result concerns a coercivity condition for scalar functions.

Lemma 30 *Let $D \subset \mathbb{R}^p$ be a nonempty bounded open set and let $\varphi : D \to \mathbb{R}$ be a continuous function and $\psi : \mathbb{R}^p \to \mathbb{R}$ be given as*

$$\psi(x) = \begin{cases} \varphi(x), & \text{if } x \in D, \\ +\infty, & \text{otherwise} \end{cases}$$

Suppose that the following condition holds: for every sequence $(x_k) \subset D$, $x_k \to \overline{x} \in \operatorname{bd} D$, the sequence $(\varphi(x_n))$ is unbounded above. Then φ and ψ are lower bounded and achieve their minimum in D.

Observe that φ_μ has the form of function ψ from Lemma 30. Moreover, since strict M is an open set, $\overline{x} \in \operatorname{bd} \operatorname{strict} M$ means that $\overline{x} \in \operatorname{cl} \operatorname{strict} M \setminus \operatorname{strict} M \subset M \setminus \operatorname{strict} M$, that is $g(\overline{x}) \in -Q \setminus - \operatorname{int} Q = - \operatorname{bd} Q$. In view of the properties of $s_{Q,c}$ this implies $s_{Q,c}(g(x)) = 0$. Therefore, $x_k \to \overline{x} \in \operatorname{bd} \operatorname{strict} M$ yields $\ln(-s_{Q,c}(g(x_k))) \to -\infty$. Consequently, under the hypotheses that X is finite dimensional, f and g are continuous and M is compact, one can apply Lemma 30 in order to deduce that there exists $x_\mu \in \operatorname{strict} M$ which is a (global) minimum for φ_μ.

Theorem 31 *Suppose all spaces are finite dimensional, M is compact,* strict $M \neq \emptyset$*, f, g are locally Lipschitz, f is $K-$convex, and g is $Q-$convex. Consider $(\mu_n) \to 0$ a sequence of positive real numbers. Then all the accumulation points of (x_{μ_n}) is a weak solution of the problem (P).*

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