Scientific activity report for the research grant

Variational approaches to set-valued optimization and applications

PN-III-P4-ID-PCE-2016-0188, nr. 10/2017

July 2017 – December 2019

The scientific activity in the period July 2017 – December 2019 was focused on the objectives of the project which are as follows:

(1) Variational principles for variable ordering structures and applications
   (1.1) Ekeland Variational Principle for variable ordering structures
   (1.2) Existence results in set-valued optimization with variable order

(2) Directional metric regularity and applications to set-valued optimization
   (2.1) Coderivative conditions for directional metric regularity
   (2.2) Optimality conditions under directional metric regularity

(3) Penalization and Aubin type properties for set-valued optimization problems
   (3.1) Vectorial penalization under vectorial Aubin conditions
   (3.2) Stability and calmness of set-valued optimization problems.

In our investigations we obtained several results within the proposed objectives and these results are contained in the papers [26], [8], [17], [10], [4], [3], [9], [2], [18], [11], [12], [45], and [46] (seven published, two accepted for publication and four submitted for publication). For more details, see

$$http://www.math.uaic.ro/\tilde{\ }zalinesc/ID-0188-en.html$$

We briefly present next the main ideas and assertions from the mentioned articles.

In the paper [26] we explore new, improved versions, of the celebrated Ekeland variational principle (EVP) which is known to be a very important tool in establishing many results in several domains of mathematics. One of its variants is the following:

Theorem 1 *Let (X,d) be a complete metric space and $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function which is bounded from below. Then for every $x_0 \in \operatorname{dom} f$ there exists $\overline{x} \in X$ such that $f(\overline{x}) + d(\overline{x}, x_0) \leq f(x_0)$ and $f(\overline{x}) < f(u) + d(u, \overline{x})$ for $u \in X \setminus \{\overline{x}\}$.*

Taking x_0 such that $f(x_0) \leq \inf f + \varepsilon$, one gets rapidly $d(\overline{x}, x_0) \leq \varepsilon$, that is $\overline{x}$, solution of the perturbed problem, is close to the ε-solution of the minimization problem min $f(x)$, $x \in X$.

Variants of EVP were established for $\overline{\mathbb{R}}$ replaced by a real separated topological vector space (tvs for short) Y quasi-ordered by a convex cone $K \subset Y$ ($y_1 \leq_K y_2$ if $y_2 - y_1 \in K$). The next step was to have EVP type results for f replaced by a set-valued function $\Gamma : X \rightrightarrows Y$ for Y a tvs quasi-ordered by the convex cone K and 2^Y quasi-ordered by $A_1 \leq_K^l A_2$ if $A_2 \subset A_1 + K$ ($A_1, A_2 \subset Y$). In this context several results were established by Ha [23], Hamel & Löhne [24], Liu & Ng [32], Qiu [36, 38], Khan, Tammer & Zalinescu [31]. Our aim is to provide a unified approach for establishing such results, getting as a by-product new sufficient conditions for having the usual conclusion.

As above, (X,d) is a metric space, Y is a tvs, Y^* is its topological dual, and $K \subseteq Y$ is a proper convex cone; as usual, K^+ is the positive dual cone of K:

$$K^+ = \{y^* \in Y^* \mid y^*(y) \geq 0 \ \forall\, y \in K\}.$$

As in [41] and [31], let $F : X \times X \rightrightarrows K$ satisfy the conditions:

(F1) $0 \in F(x,x)$ for all $x \in X$,

(F2) $F(x_1,x_2) + F(x_2,x_3) \subseteq F(x_1,x_3) + K$ for all $x_1, x_2, x_3 \in X$.

For F satisfying conditions (F1) and (F2), and $z^* \in K^+$, consider

$$\eta_{F,z^*} : X \times X \to \overline{\mathbb{R}}_+, \quad \eta_{F,z^*}(x,x') := \inf\{z^*(z) \mid z \in F(x,x')\}.$$

It follows immediately that

$$\eta_{F,z^*}(x,x) = 0 \text{ and } \eta_{F,z^*}(x,x'') \le \eta_{F,z^*}(x,x') + \eta_{F,z^*}(x',x'') \quad \forall x,x',x'' \in X.$$

Using F we introduce the quasi-order $\preceq_F$ on $X \times 2^Y$ defined by

$$(x_1,A_1) \preceq_F (x_2,A_2) :\iff A_2 \subset A_1 + F(x_1,x_2) + K. \tag{1}$$

Of course,

$$(x_1,A_1) \preceq_F (x_2,A_2) \Rightarrow A_2 \subset A_1 + K \iff: A_1 \le^l_K A_2;$$

moreover, by (F1), we have that

$$(x,A_1) \preceq_F (x,A_2) \iff A_2 \subset A_1 + K \iff A_1 \le^l_K A_2.$$

Besides (F1) and (F2) we consider also the condition

(F3) there exists $z^*_F \in K^+$ such that

$$\eta(\delta) := \inf z^*_F(F_\delta) := \inf\{z^*_F(v) \mid v \in F_\delta\} > 0 \quad \forall \delta > 0,$$

where, for $\delta \ge 0$,

$$F_\delta := \cup\{F(x,x') \mid x,x' \in X,\ d(x,x') \ge \delta\};$$

clearly, condition (F3) can be rewritten as

$$\exists z^*_F \in K^+,\ \forall \delta > 0 \ :\ \inf z^*_F(F_\delta) > 0.$$

A weaker condition is

$$\forall \delta > 0,\ \exists z^* \in K^+ \ :\ \inf z^*(F_\delta) > 0. \tag{2}$$

An even weaker condition is the following

$$\forall \delta > 0,\ \forall (z_n) \subseteq F_\delta,\ \exists z^* \in K^+ \ :\ \limsup z^*(z_n) > 0; \tag{3}$$

when (3) holds Qiu [37, Def. 3.5] says that F *is compatible with* d.

An important example of multifunction F satisfying conditions (F1) and (F2) is provided in the next result (proved in [31, Lem. 10.1.1]).

Lemma 2 *Let $\emptyset \ne H \subseteq K$ be a K–convex set. Consider*

$$F_H : X \times X \rightrightarrows K, \quad F_H(x,x') := d(x,x')H.$$

Then

- (i) *F_H verifies (F1) and (F2).*
- (ii) *F_H verifies condition (F3) iff F_H verifies condition (2) iff there exists $z^*_H \in K^+$ such that $\inf z^*_H(H) > 0$; if Y is a separated locally convex space, then F_H verifies condition (F3) iff $0 \notin \operatorname{cl}(H+K)$. Moreover, F_H verifies condition (3) iff*
$$\forall (h_n) \subseteq H,\ \exists z^* \in K^+ \ :\ \limsup z^*(h_n) > 0.$$

For F satisfying conditions (F1) and (F2), and $z^* \in K^+$, we introduce the partial order $\preceq_{F,z^*}$ on $X \times 2^Y$ by

$$(x_1,A_1) \preceq_{F,z^*} (x_2,A_2) :\iff \begin{cases} (x_1,A_1) = (x_2,A_2) \text{ or} \\ (x_1,A_1) \preceq_F (x_2,A_2) \text{ and } \inf z^*(A_1) < \inf z^*(A_2). \end{cases} \tag{4}$$

It is easy to verify that $\preceq_{F,z^*}$ is, indeed, reflexive, transitive, and antisymmetric. We denote by $\preceq_H$ (resp. $\preceq_{H,z^*}$) the partial order $\preceq_{F_H}$ (resp. $\preceq_{F_H,z^*}$) when $H \subset K$ is a K-convex set.

Below (X,d) is a metric space, Y is a separated topological vector space, $K \subset Y$ is a proper convex cone, and $F : X \times X \rightrightarrows K$ verifies conditions (F1) and (F2). On $X \times 2^Y$ we consider the quasi-order $\preceq_F$, as well as $\preceq_{F,z^*}$ for $z^* \in K^+$, defined in (1) and (4), respectively.

Moreover, we consider a nonempty set $\mathcal{A} \subseteq X \times 2^Y$. Because $(x,A) \preceq_F (x',\emptyset)$ for all $x,x' \in X$ and $A \in 2^Y$, in the sequel we assume that $A \neq \emptyset$ for every $A \in \mathrm{Pr}_{2^Y}(\mathcal{A})$; hence

$$Y_{\mathcal{A}} := \bigcup \{A \mid A \in \mathrm{Pr}_{2^Y}(\mathcal{A})\} \neq \emptyset.$$

An important example of set $\mathcal{A} \subseteq X \times 2^Y$ is

$$\mathcal{A}_\Gamma := \{((x,\Gamma(x))) \mid x \in \mathrm{dom}\,\Gamma\},$$

where $\Gamma : X \rightrightarrows Y$ with $\mathrm{dom}\,\Gamma \neq \emptyset$; of course, $Y_{\mathcal{A}_\Gamma} = \Gamma(X)$.

Ha [23] established an EVP type result for a set-valued function $\Gamma : X \rightrightarrows Y$ which corresponds to Kuroiwa optimality. Hamel [25] and Hamel–Löhne [24] established such results for subsets $\mathcal{A} \subseteq X \times 2^Y$ even for X a uniform space.

Theorem 3 *Assume that $(\mathcal{A}, \preceq_F)$ verifies condition*

(C0) $\forall\, ((x_n, A_n))_{n\geq 1} \subset \mathcal{A}$ *$\preceq_F$-decreasing : $(x_n)_{n\geq 1}$ is Cauchy and $\exists (x,A) \in \mathcal{A}$ such that $(x,A) \preceq (x_n,A_n)$ $\forall n \geq 1$.*

Then:

(i) *for every $(x,A) \in \mathcal{A}$ there exists $(\overline{x},\overline{A}) \in \mathcal{A}$ such that $(\overline{x},\overline{A}) \preceq_F (x,A)$ and $\mathcal{A} \ni (x',A') \preceq_F (\overline{x},\overline{A})$ implies $x' = \overline{x}$;*

(ii) *assume that $z^* \in K^+$ is such that $\inf z^*(A) > -\infty$ for $A \in \mathrm{Pr}_{2^Y}(\mathcal{A})$ and $\inf z^*(F(x,x')) > 0$ for $x,x' \in X$ with $x \neq x'$; then for every $(x,A) \in \mathcal{A}$ there exists $(\overline{x},\overline{A}) \in \mathcal{A}$ minimal with respect to $\preceq_{F,z^*}$ such that $(\overline{x},\overline{A}) \preceq_{F,z^*} (x,A)$ and $\mathcal{A} \ni (x',A') \preceq_F (\overline{x},\overline{A})$ implies $x' = \overline{x}$.*

Remark 4 *Note that, for having the conclusions of Theorem 3 (i) or (ii) only for a given $(x_0,A_0) \in \mathcal{A}$, it is sufficient to assume that (C0) is verified by the sets*

$$\mathcal{A}_F(x_0,A_0) := \{(x,A) \in \mathcal{A} \mid (x,A) \preceq_F (x_0,A_0)\},$$
$$\mathcal{A}_{F,z^*}(x_0,A_0) := \{(x,A) \in \mathcal{A} \mid (x,A) \preceq_{F,z^*} (x_0,A_0)\},$$

respectively.

Remark 5 *Taking $F(x,x') := \{d(x,x')k^0\}$ with $k^0 \in K \setminus (-\mathrm{cl}\, K)$ in Theorem 3 (using also Remark 4) one obtains [24, Th. 5.1] in the case X is a metric space. Indeed, on $\mathcal{A}_0$, if (M2) or (M2') is verified, then any $\preceq_F$–decreasing sequence in $\mathcal{A}$ is Cauchy. This together with (M3) shows that (C0) holds. In a similar way [24, Th. 6.1] can be obtained.*

Remark 6 *Taking X, Y, K, H and $\Gamma : X \rightrightarrows Y$ defined in Example 7 below, $\mathcal{A}_\Gamma$ verifies the conditions*

(C1) $\forall\, ((x_n, A_n))_{n\geq 1} \subset \mathcal{A}$ $\preceq$-decreasing with $(x_n)_{n\geq 1}$ Cauchy : $\exists (x,A) \in \mathcal{A}$ such that $(x,A) \preceq (x_n,A_n)$ $\forall n \geq 1$,

(C'1) $\forall\, ((x_n, z_n))_{n\geq 1} \subset \mathcal{A}$ $\preceq$-decreasing with $x_n \to x \in X$: $\exists z \in Z$ such that $(x,z) \in \mathcal{A}$ and $(x,z) \preceq (x_n,z_n)$ $\forall n \geq 1$,

for $F := F_H$, but not (C0). Moreover, the conclusion of Theorem 3 (i) does not hold. This shows that, in order to have the conclusion of Theorem 3 we need supplementary conditions besides (C'1) or (C1).

Example 7 *Let $X := \mathbb{R}$ and $Y := \mathbb{R}^2$ be endowed with their usual norms, $K := \mathbb{R} \times \mathbb{R}_+$, $H := \{(y_1,y_2) \in \mathbb{R}^2_+ \mid y_1y_2 \geq 1\}$, and $\Gamma : X \rightrightarrows Y$, $\Gamma(x) := \{(x,e^x)\}$. It is clear that H is a closed convex subset of $K \setminus \{(0,0)\}$ and $K + \varepsilon H = \mathrm{int}\, K = \mathbb{R} \times \mathbb{R}^*_+$ for $\varepsilon > 0$, where $\mathbb{R}^*_+ := \mathbb{R}_+ \setminus \{0\}$. One has $\Gamma(x) + K = \mathbb{R} \times [e^x, \infty)$, and so, for $x,x' \in X$ and $\alpha > 0$,*

$$\Gamma(x) \preceq_H \Gamma(x') \Leftrightarrow x \leq x' \Leftrightarrow \Gamma(x) \leq^l_K \Gamma(x'), \quad \Gamma(x') \subset \Gamma(x) + \alpha H + K \Leftrightarrow x < x';$$

moreover, $\Gamma(X) = \{(x,e^x) \mid x \in \mathbb{R}\} \subset K$, which shows that $\Gamma(X)$ is K–bounded. So, for the sequence $(x_n)_{n\geq 1} \subset X$ with $x_n \to x$ and $\Gamma(x_n) \subset \Gamma(x_{n+1}) + K$ (that is $\Gamma(x_{n+1}) \leq^l_K \Gamma(x_n)$) for $n \geq 1$, we have that $x_{n+1} \leq x_n$, and so $x \leq x_n$ for $n \geq 1$, whence $\Gamma(x_n) \subset \Gamma(x) + K$ for $n \geq 1$. This shows that (C'1) *and* (C1) *are verified; however, taking $x_n := -n$ (for $n \geq 1$) it is clear that* (C0) *is not verified.*

Remark 8 *Let $\Gamma : X \rightrightarrows Y$ have nonempty domain. Set $x \preceq u$ if $(x, \Gamma(x)) \preceq_F (u, \Gamma(u))$. Note that $\mathcal{A}_\Gamma$ verifies condition (C'1) if and only if $S(u) := \{x \in X \mid x \preceq u\}$ is $\preceq$–lower closed for every $u \in X$. This shows that condition (C'1) extends the dynamic closedness of a set-valued mapping as defined in [38] and elsewhere.*

Theorem 9 *Assume that the following two conditions hold:*

(i) *F verifies conditions* (F1), (F2) *and* (F3)*;*

(ii) *$\mathcal{A}$ verifies* (C1) *and $z_F^*(Y_\mathcal{A})$ is bounded from below, where $z_F^* \in K^+$ is provided by* (F3)*.*

Then for every $(x, A) \in \mathcal{A}$ there exists a minimal element $(\overline{x}, \overline{A}) \in \mathcal{A}$ with respect to $\preceq_{F,z_F^*}$ such that $(\overline{x}, \overline{A}) \preceq_{F,z_F^*} (x, A)$; moreover $\mathcal{A} \ni (x', A') \preceq_F (\overline{x}, \overline{A})$ implies $x' = \overline{x}$.

In the case in which $F = F_H$ with $H \subset K$ a K-convex set, we provide several sufficient conditions for having the conclusion of the preceding theorem; these cover the majority of the EVP type results for set-valued mappings found in the literature.

In the paper [10] we give new insight to some questions raised by Lasserre in [29] and [30], concerning the preservation of the necessary and sufficient optimality conditions from smooth convex optimization with inequalities constraints to the case where the feasible set is convex, but has no convex representation. The main results we obtain concerns, some relations between the hypotheses imposed by Lasserre and the Mangasarian-Fromowitz condition, and a barrier method based only on the geometric representation of the feasible set. Let X be a normed vector space and $f : X \to \mathbb{R}$ be a real-valued function. Take $\emptyset \neq M \subset X$ be a closed convex set and consider the standard geometrically constrained optimization problem

$$(P)\ \ \min f(x),\ \text{ s.t. } x \in M.$$

If f is convex and continuous, the Pshenichnyi-Rockafellar Theorem says that $\overline{x} \in M$ is a solution of (P) if and only if

$$\partial f(\overline{x}) \cap -N(M, \overline{x}) \neq \emptyset, \tag{5}$$

where ∂ denotes the convex subdifferential and N stands for the convex normal cone, that is

$$N(M, \overline{x}) := \{x^* \in X^* \mid \langle x^*, x - \overline{x}\rangle \leq 0, \forall x \in M\}.$$

Some interesting facts were derived in [29], [30] for the case when M is defined by functional inequalities, that is,

$$M := \{x \in X \mid g_i(x) \leq 0,\ \forall i \in \overline{1, n}\}, \tag{6}$$

where n is a nonzero natural number, and $g_i : X \to \mathbb{R}$ are scalar, not necessarily convex functions (that means one has a nonconvex representation for the convex set M). As usual, for $x \in M$ one denotes by $I(x)$ the set of active indices at x, that is, $I(x) := \{i \in \overline{1, n} \mid g_i(x) = 0\}$.

In order to present Karush-Kuhn-Tucker necessary and sufficient condition for this situation and the convergence of a barrier method in this setting, the works [29] and [30] use the following two assumptions in order get the results, that are:

- the Slater condition for M (there exists $x_0 \in M$ such that for all $i \in \overline{1, n}$, $g_i(x_0) < 0$) and
- the non-degeneracy condition: for every $i \in \overline{1, n}$,

$$x \in M, g_i(x) = 0 \Longrightarrow \nabla g_i(x) \neq 0. \tag{7}$$

We obtained the following results concerning the link between these conditions and the celebrated Mangasarian-Fromowitz constraint qualification condition.

Proposition 10 *Suppose that X is a Banach space, and $M \subset X$ is closed and convex. Take $x \in M$. Then the non-degeneracy condition at x and the Slater condition hold if and only if the Mangasarian-Fromowitz constraint qualification condition at x holds (that is, there exists $u \in X$ such that for every $i \in I(\overline{x})$ one has $\langle \nabla g_i(\overline{x}), u\rangle < 0$).*

Proposition 11 *If X is a Banach space and $M \subset X$ is closed and convex, under non-degeneracy condition, the Slater condition is equivalent to* $\operatorname{int} M \neq \emptyset$.

Next, we propose a barrier method for geometrically constrained problem (P) in the convex case (that is, f and M are convex) and we show its convergence to a minimum point. To this aim we use the oriented distance function associated to set $A \subset X$, $\Delta_A : X \to \mathbb{R}$, given as

$$\Delta_A(y) := d_A(y) - d_{Y \setminus A}(y).$$

Let $f : X \to \mathbb{R}$ be a continuous function and $\emptyset \neq M \subset X$ be a compact set with nonempty interior. Consider again the constrained optimization problem

$$(P) \ \min f(x), \text{ s.t. } x \in M.$$

Take $\mu > 0$ and define $\varphi_\mu : X \to \mathbb{R} \cup \{+\infty\}$ as

$$\varphi_\mu(x) = \begin{cases} f(x) - \mu \ln(-\Delta_M(x)), & \text{if } x \in \operatorname{int} M, \\ +\infty, & \text{otherwise.} \end{cases} \tag{8}$$

Observe that, in view of the properties of Δ_M, the function φ_μ is well defined.

Further, consider that $X = \mathbb{R}^p$ $(p \geq 1)$ and let $(x_n) \subset \operatorname{int} M$ be a sequence convergent to $x \in \operatorname{bd} M$. Then, from the continuity of Δ_M and f, one has that $\Delta_M(x_n) \to 0$ and $f(x_n) - \mu \ln(-\Delta_M(x_n)) \to +\infty$. This means that there is $x_\mu \in \operatorname{int} M$ which is a global minimum of φ_μ. Using the results in the previous results, under suitable qualification conditions, the point x is a Karush-Kuhn-Tucker point for (P). We prove the following results.

Theorem 12 *Suppose that $X = \mathbb{R}^p$ $(p \geq 1)$, f is a convex function, and M is a convex and compact set. Then for each $\mu > 0$, there exists a global minimizer x_μ of φ_μ in $\operatorname{int} M$, and if $\mu_n \downarrow 0$, every accumulation point of $\{x_{\mu_n}\}$ is a global minimizer of (P).*

Theorem 13 *Suppose that $X = \mathbb{R}^p$ $(p \geq 1)$, f is a convex function, and M given by (6) is a convex and compact set. If Slater and non-degeneracy conditions hold, then for each $\mu > 0$ there exists a global minimizer x_μ of φ_μ in $\operatorname{int} M$, and if $\mu_n \downarrow 0$, every accumulation point of $\{x_{\mu_n}\}$ is a global minimizer of (P) that satisfies Karush-Kuhn-Tucker conditions.*

The paper [11] continues the investigation from [10] and we present a barrier method for vector optimization problems with inequality constraints. To this aim, we firstly investigate some constraint qualification conditions and we compare them to the corresponding ones in literature. Then, we define a barrier function and observe that its basic properties do work for fairly general situations, while for meaningful convergence results of the associated barrier method we should restrict ourselves to convex case and finite dimensional setting.

Let X, Y, Z be Banach spaces and let us consider on Y and Z some partial order relations given by the closed convex and pointed cones $K \subset Y$ and $Q \subset Z$, respectively. More precisely, on Y we have the relation $\leq_K$ given by the equivalence $y_1 \leq_K y_2$ iff $y_2 - y_1 \in K$ and, similarly, the relation $\leq_Q$ on Z. Moreover, we suppose that $\operatorname{int} K \neq \emptyset$ and $\operatorname{int} Q \neq \emptyset$.

Take $f : X \to Y$ and $g : X \to Z$ as vectorial continuous single-valued mappings and consider the following vector optimization problem:

$$(P) \ \min f(x) \text{ s.t. } g(x) \in -Q.$$

Here, in view of the assumptions on the cone K, the optimality is understood in the weak Pareto sense: $\overline{x}$ is a weak solution of the problem (P) if for any x satisfying the constraint (i.e., $g(x) \in -Q$), $f(x) - f(\overline{x}) \notin -\operatorname{int} K$. Observe that the restriction $g(x) \in -Q$ is a generalized inequality constraint, since in the situation $Z := \mathbb{R}^p$, $Q := \mathbb{R}^p_+ = [0, \infty)^p$ this reduces to the system of inequalities $g_i(x) \leq 0$ for all $i \in \overline{1,p}$ where g_i are the coordinates functions of g.

We aim at introducing a barrier function associated to (P) and for that, first of all, we have to consider the set M of feasible points, that is, $M := \{x \in X \mid g(x) \in -Q\}$, the set $\operatorname{int} M$ and the set $\operatorname{strict} M := \{x \in X \mid g(x) \in -\operatorname{int} Q\}$. In view of the continuity of g and the closedness of Q, the set M is closed, while the sets $\operatorname{int} M$, $\operatorname{strict} M$ are obviously open. Clearly, $\operatorname{strict} M \subset \operatorname{int} M$. In some situations, for a proper definition of the barrier function we envisage, one needs to have the equality $\operatorname{strict} M \subset \operatorname{int} M$ which, in general, does not hold. The fact that $\operatorname{strict} M \neq \emptyset$ is nothing else but the well-known Slater condition: there exists $x \in X$ with $g(x) \in -\operatorname{int} Q$.

In order to formulate the result that ensures the equality between $\operatorname{int} M$ and $\operatorname{strict} M$, we need the following definition: one says that g is open at a point $x \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $B(g(x), \delta) \subset g(B(x, \varepsilon))$.

Lemma 14 *Suppose that g is open at every point x for which $g(x) \in -\operatorname{bd} Q$. Then* $\operatorname{strict} M = \operatorname{int} M$.

Another preparatory discussion concern the main tool we use to scalarize both the objective function and the constraint system of problem (P). More precisely, we use the Gerstewitz functional in the special case when the ordering cone has nonempty interior.

Theorem 15 *Let $K \subset Y$ be a closed convex cone with nonempty interior. Then for every $e \in \operatorname{int} K$ the functional $s_{K,e} : Y \to \mathbb{R}$ given by*

$$s_{K,e}(y) = \inf\{\lambda \in \mathbb{R} \mid \lambda e \in y + K\} \tag{9}$$

is convex continuous and for every $\lambda \in \mathbb{R}$,

$$\{y \in Y \mid s_{K,e}(y) < \lambda\} = \lambda e - \operatorname{int} K, \text{ and } \{y \in Y \mid s_{K,e}(y) = \lambda\} = \lambda e - \operatorname{bd} K. \tag{10}$$

Moreover, $s_{K,e}$ is sublinear, $K-$monotone (that is, for all $y_1, y_2 \in Y$, $y_1 \leq_K y_2$ implies $f(y_1) \leq f(y_2)$), strictly-$\operatorname{int} K$*-monotone (that is, for all $y_1, y_2 \in Y$, $y_2 - y_1 \in \operatorname{int} K$ implies $f(y_1) < f(y_2)$) and for every $u \in Y$, the Fenchel (convex) subdifferential $\partial s_{K,e}(u)$ is nonempty and*

$$\partial s_{K,e}(u) = \{v^* \in K^* \mid v^*(e) = 1, v^*(u) = s_{K,e}(u)\}.$$

In addition, $s_{K,e}$ is $d(e, \operatorname{bd}(K))^{-1}$–Lipschitzian.

The next lemma links the minima of the composite function $s_{K,e} \circ f$ with the minimizers of f.

Lemma 16 *If $\overline{x} \in M$ is a minimum on M of the scalar function $s_{K,e} \circ f$, then it is a weak solution of the problem (P) as well.*

Another useful result concerns a coercivity condition for scalar functions.

Lemma 17 *Let $D \subset \mathbb{R}^p$ be a nonempty bounded open set and let $\varphi : D \to \mathbb{R}$ be a continuous function and $\psi : \mathbb{R}^p \to \mathbb{R}$ be given as*

$$\psi(x) = \begin{cases} \varphi(x), & \text{if } x \in D, \\ +\infty, & \text{otherwise} \end{cases}$$

Suppose that the following condition holds: for every sequence $(x_k) \subset D$, $x_k \to \overline{x} \in \operatorname{bd} D$, the sequence $(\varphi(x_n))$ is unbounded above. Then φ and ψ are lower bounded and achieve their minimum in D.

Observe that φ_μ has the form of function ψ from Lemma 17. Moreover, since $\operatorname{strict} M$ is an open set, $\overline{x} \in \operatorname{bd} \operatorname{strict} M$ means that $\overline{x} \in \operatorname{cl} \operatorname{strict} M \setminus \operatorname{strict} M \subset M \setminus \operatorname{strict} M$, that is $g(\overline{x}) \in -Q \setminus -\operatorname{int} Q = -\operatorname{bd} Q$. In view of the properties of $s_{Q,c}$ this implies $s_{Q,c}(g(x)) = 0$. Therefore, $x_k \to \overline{x} \in \operatorname{bd} \operatorname{strict} M$ yields $\ln(-s_{Q,c}(g(x_k))) \to -\infty$. Consequently, under the hypotheses that X is finite dimensional, f and g are continuous and M is compact, one can apply Lemma 17 in order to deduce that there exists $x_\mu \in \operatorname{strict} M$ which is a (global) minimum for φ_μ.

Theorem 18 *Suppose all spaces are finite dimensional, M is compact,* $\operatorname{strict} M \neq \emptyset$*, f, g are locally Lipschitz, f is $K-$convex, and g is $Q-$convex. Consider $(\mu_n) \to 0$ a sequence of positive real numbers. Then all the accumulation points of (x_{μ_n}) is a weak solution of the problem (P).*

In the paper [8] we introduced a notion of Henig proper efficiency for constrained vector optimization problems in the setting of variable ordering structure. In order to get an appropriate concept, we had to explore firstly the case of fixed ordering structure and to observe that, in certain situations, the well-known Henig proper efficiency can be expressed in a simpler way. Then, we observe that the newly introduced notion can be reduced, by a Clarke-type penalization result, to the notion of unconstrained robust efficiency.

We show that this penalization technique, coupled with sufficient conditions for weak openness, serves as a basis for developing necessary optimality conditions for our Henig proper efficiency in terms of generalized differentiation objects lying in both primal and dual spaces.

More precisely, the setting of this work is as follows. Let X, Y be normed vector spaces. For $x \in X$ and $r > 0$, denote by $B_X(x,r)$, $D_X(x,r)$ and $S_X(x,r)$ the open and the closed balls, and the sphere of center x and radius r, respectively. In the case where $x := 0$ and $r := 1$, we use the notations B_X, D_X and S_X. For $x \in X$, the symbol $\mathcal{V}(x)$ stands for the system of neighborhoods of x. For a set $A \subset X$, we denote by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$ its topological interior, closure and boundary, respectively. The cone generated by A is designated by $\operatorname{cone} A$, and the convex hull of A is $\operatorname{conv} A$. The distance from a point x to a set A is $d(x,A) := \inf \{\|x-a\| \mid a \in A\}$. The notation X^* stands for the topological dual of X. On a product space we consider the sum norm, unless otherwise stated.

Let $F : X \rightrightarrows Y$ be a multifunction. The graph of F is denoted by $\operatorname{Gr} F = \{(x,y) \in X \times Y \mid y \in F(x)\}$. If $A \subset X$, then $F(A) := \bigcup_{x \in A} F(x)$ and the inverse set-valued mapping of F is $F^{-1} : Y \rightrightarrows X$ given by $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$.

Consider $\emptyset \neq M \subset X$ as a closed set and $C \subset Y$ as a closed convex pointed cone. We recall that a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$ is an efficient point for F on M with respect to C if there is a neighborhood U of $\overline{x}$ such that for every $x \in M \cap U$ one has

$$(F(x) - \overline{y}) \cap (-C) \subset \{0\}.$$

Denote by $\operatorname{Eff}(F, M; C)$ the set of efficient points for F on M with respect to C.

If $\operatorname{int} C \neq \emptyset$, a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$ is a weakly efficient point for F on M with respect to C if there is a neighborhood U of $\overline{x}$ such that for every $x \in M \cap U$ one has

$$(F(x) - \overline{y}) \cap (-\operatorname{int} C) = \emptyset,$$

and we denote by $\operatorname{WEff}(F, M; C)$ the set of weakly efficient points for F on M with respect to C.

Now, following [22, p. 110], the set of Henig-proper efficient points of F on M with respect to C is

$$\operatorname{HEff}(F, M; C) = \bigcup \{\operatorname{Eff}(F, M; K) \mid K \text{ convex cone, } C \setminus \{0\} \subset \operatorname{int} K \neq Y\}.$$

When $M = X$, then we have unconstrained efficiencies and we drop the notation M in the writing of efficiency sets.

A convex set B is said to be a base for the cone C if $0 \notin \operatorname{cl} B$ and $C = \operatorname{cone} B$. A cone which admits a base is called based. In this case one can consider the so-called Henig dilating cones C_ε, $\varepsilon \in (0, d(0,B))$ for which the definition and properties are presented, for instance, in [22, Lemma 3.2.51].

The next cone separation result is obtained in [7].

Theorem 19 *Let $P, Q \subset Y$ be closed cones such that Q admits a closed base B. Moreover, suppose that P or B is a.c. If $P \cap Q = \{0\}$, then there exists $\varepsilon \in]0, d(0,B)[$ such that $P \cap Q_\varepsilon = \{0\}$.*

Now, if one supposes that C has a closed asymptotically compact (a.c., for short) base B and take, as above, $\delta = d(0,B)$, then we can write the following relation:

$$\operatorname{HEff}(F, M; C) = \bigcup \{\operatorname{Eff}(F, M; C_\varepsilon) \mid \varepsilon \in]0, \delta[\}.$$

This remark is essential in what follows.

Consider $K : X \rightrightarrows Y$ a multifunction whose values are proper pointed closed convex cones. This leads us, for every $x \in X$, to an order relation on Y : $y_1 \leq_{K(x)} y_2 \Leftrightarrow y_2 - y_1 \in K(x)$. Suppose that all cones $K(x)$ are based. On the basis of the previous discussion, we define the set of Henig-proper nondominated points of F on M with respect to K as

$$\begin{aligned} \operatorname{VosHEff}(F, M; K) = \{(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (M \times Y) \mid \exists \varepsilon > 0, \exists U \in \mathcal{V}(\overline{x}), \forall x \in U \cap M, \\ (F(x) - \overline{y}) \cap (-K(x)_\varepsilon) \subset \{0\}\}. \end{aligned} \tag{11}$$

This newly introduced concept clearly covers the case of Henig proper efficiency described before provided that the base of $K(x)$ with x close to $\overline{x}$ is a.c., and, moreover, this is in the same line with other notions of efficiency developed in [15] and [16]. We give a short account on these concepts, since we investigate in the sequel some interesting links between all these efficiencies.

According to [15], a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$ is a (local) nondominated point for F on M with respect to K if there is a neighborhood U of $\overline{x}$ such that for every $x \in M \cap U$, one has

$$(F(x) - \overline{y}) \cap (-K(x)) \subset \{0\}.$$

If $\operatorname{int} K(x) \neq \emptyset$ for every x in a neighborhood V of $\overline{x}$, a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$ is a weakly nondominated point for F on M with respect to K if there is a neighborhood $U \subset V$ of $\overline{x}$ such that for every $x \in M \cap U$, one has

$$(F(x) - \overline{y}) \cap (-\operatorname{int} K(x)) = \emptyset.$$

As a natural generalization of the above notions, the following concepts of robustness were introduced in [16].

A point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$ is a local robust efficient point for F on M with respect to K if there is a neighborhood U of $\overline{x}$ such that for every $x, z \in U \cap M$, one has

$$(F(x) - \overline{y}) \cap (-K(z)) \subset \{0\}.$$

If $\operatorname{int} K(z) \neq \emptyset$ for every z in a neighborhood V of $\overline{x}$, then one says that a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$ is a local robust weakly efficient point for F on M with respect to K if there is a neighborhood $U \subset V$ of $\overline{x}$ such that for every $x, z \in U \cap M$, one has

$$(F(x) - \overline{y}) \cap (-\operatorname{int} K(z)) = \emptyset.$$

When $M = X$, we have unconstrained efficiencies and we omit to write "on M".

In order to investigate the constrained Henig-proper nondomination from the perspective of necessary optimality conditions, we firstly need to reduce this constrained efficiency to an unconstrained one. As usual, this can be done by a penalization result and this task is accomplished in the next theorem which is in the line of Clarke penalization method, that is, asks for some local Lipschitz behavior of the involved multifunctions and uses the distance to the set of feasible points M as a penalty term. Moreover, we prove that a $\operatorname{VosHEff}(F, M; K)$ point is reduced in this way to an unconstrained robust efficient point.

In the first penalization result, we consider the situation when the ordering cones have nonempty topological interior, that is the setting known in vector optimization as the weak case.

Theorem 20 *Let* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$. *Suppose that there exists a neighborhood* U *of* $\overline{x}$ *such that:*
(i) $\operatorname{int} K(x) \neq \emptyset$ *for all* $x \in U$ *and there exists* $e \in Y \setminus \{0\}$ *such that* $e \in K(x)$ *for all* $x \in U$;
(ii) there exists a constant $L > 0$ *such that for all* $u, v \in U$,

$$F(u) \subset F(v) - L \|u - v\| e + K(v);$$

(iii) $K(x)$ *is based for all* $x \in U$ *(the base of* $K(x)$ *is denoted by* $B(x)$*);*
(iv) there exists a constant $l > 0$ *such that for all* $u, v \in U$,

$$B(u) \subset B(v) + l \|u - v\| D_Y.$$

If $(\overline{x}, \overline{y}) \in \operatorname{VosHEff}(F, M; K)$, *then* $(\overline{x}, \overline{y})$ *is a local robust weakly efficient point for* $F(\cdot) + L d_M(\cdot) e$ *with respect to* K.

Now we do not suppose that $\operatorname{int} K(x) \neq \emptyset$ and then we get a result for the strong case.

Theorem 21 *Let* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$. *Suppose that there exists a neighborhood* U *of* $\overline{x}$ *such that:*
(i) there exists $e \in Y \setminus \{0\}$ *such that* $e \in K(x)$ *for all* $x \in U$;
(ii) there exists a constant $L > 0$ *such that for all* $u, v \in U$,

$$F(u) \subset F(v) - L \|u - v\| e + K(v);$$

(iii) $K(x)$ is based for all $x \in U$ (the base of $K(x)$ is denoted by $B(x)$);

(iv) there exists a constant $l > 0$ such that for all $u, v \in U$,

$$B(u) \subset B(v) + l \|u - v\| D_Y.$$

If $(\overline{x}, \overline{y}) \in \mathrm{VosHEff}(F, M; K)$, then for all $\beta > L$, $(\overline{x}, \overline{y})$ is a local robust efficient point for $F(\cdot) + \beta d_M(\cdot)e$ with respect to K.

Let $F,\ K : X \rightrightarrows Y$ be the multifunctions considered previously. We aim to get necessary optimality conditions for Henig-proper nondominated points and in order to do this, according to the penalty results from the previous section (that are, Theorems 20 and 21), we need to provide appropriate necessary optimality conditions for the robust efficiency for the sum of two mappings (the objective F and the penalty term).

Recall (see [16]) that the pair (F, K) is weakly open at $(\overline{x}, \overline{y})$, or $F + K$ is weakly open at $(\overline{x}, \overline{y}) \in \mathrm{Gr}(F + K)$ if for every neighborhood U of $\overline{x}$, there exists a neighborhood V of $\overline{y}$ such that $V \subset F(U) + K(U)$. Motivated by the incompatibility proved in [16] between weak openness and the robust efficiency, and by the fact that the penalty results presented before involve three mappings, in this section we obtain sufficient conditions for the weak openness of pairs of the form $(F + G, K)$, where $G : X \rightrightarrows Y$ is a multifunction.

We denote by (F, G) the multifunction $(F, G) : X \rightrightarrows Y \times Y$ given by $(F, G)(x) := F(x) \times G(x)$ for every $x \in X$, and by (F, G, K) the multifunction $(F, G, K) : X \rightrightarrows Y \times Y \times Y$ given by $(F, G, K)(x) := F(x) \times G(x) \times K(x)$ for every $x \in X$.

Recall the main generalized differentiation objects defined on primal spaces. The definitions are standard (see, for instance, [1]).

Definition 22 *Let S be a nonempty subset of X and $\overline{x} \in X$.*

(i) The Bouligand tangent cone to S at $\overline{x}$ is the set

$$T_B(S, \overline{x}) = \{u \in X \mid \exists (t_n) \downarrow 0, \exists (u_n) \to u, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \overline{x} + t_n u_n \in S\},$$

where $(t_n) \downarrow 0$ means $(t_n) \subset]0, \infty[$ and $(t_n) \to 0$.

(ii) The Ursescu tangent cone to S at $\overline{x}$ is the set

$$T_U(S, \overline{x}) = \{u \in X \mid \forall (t_n) \downarrow 0, \exists (u_n) \to u, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \overline{x} + t_n u_n \in S\}.$$

Based on these concepts, one defines the associated derivatives for set-valued maps.

Definition 23 *Let $(\overline{x}, \overline{y}) \in \mathrm{Gr}\, F$. The Bouligand derivative of F at $(\overline{x}, \overline{y})$ is the set valued map $D_B F(\overline{x}, \overline{y})$ from X into Y defined by*

$$\mathrm{Gr}\, D_B F(\overline{x}, \overline{y}) = T_B(\mathrm{Gr}\, F, (\overline{x}, \overline{y})).$$

Note that the Ursescu derivative, denoted by $D_U F(\overline{x}, \overline{y})$, has a similar definition. One says that a set-valued map F is proto-differentiable at $\overline{x}$ relative to $\overline{y} \in F(\overline{x})$ if $D_U F(\overline{x}, \overline{y}) = D_B F(\overline{x}, \overline{y})$ (see [40]). Finally, the set-valued mapping F is said to have the Aubin property around $(\overline{x}, \overline{y})$ with constant $M > 0$ if there exist a neighborhood U of $\overline{x}$ and a neighborhood V of $\overline{y}$ such that, for every $x, u \in U$,

$$F(x) \cap V \subset F(u) + M \|x - u\| D_Y.$$

The first weak openness result, with the assumption in terms of derivatives, is given in the next theorem.

Theorem 24 *Let X, Y be Banach spaces, $F,\ G,\ K : X \rightrightarrows Y$ be multifunctions such that $\mathrm{Gr}\, F$, $\mathrm{Gr}\, G$, $\mathrm{Gr}\, K$ are closed around $(\overline{x}, \overline{y}) \in \mathrm{Gr}\, F$, $(\overline{x}, \overline{w}) \in \mathrm{Gr}\, G$ and $(\overline{x}, \overline{z}) \in \mathrm{Gr}\, K$, respectively. Suppose, moreover, that there exists $\lambda > 0$ such that, for every $(x, y, w, t, z) \in \mathrm{Gr}\,(F, G) \times \mathrm{Gr}\, K$ around $(\overline{x}, \overline{y}, \overline{w}, \overline{x}, \overline{z})$, the following assumptions are satisfied:*

(i) the next inclusion holds

$$B_Y(0, \lambda) \subset \mathrm{cl}[(D_B F(x, y) + D_B G(x, w) \cap B_Y(0, 1))\,(B_X(0, 1)) + D_B K(t, z)\,(B_X(0, 1)) \cap B_Y(0, 1)]; \tag{12}$$

(ii) either G is proto-differentiable at x relative to w and K is proto-differentiable at t relative to z, or G is proto-differentiable at x relative to w and F is proto-differentiable at x relative to y, or F is proto-differentiable at x relative to y and K is proto-differentiable at t relative to z;

(iii) either F has the Aubin property around the point (x,y), or G has the Aubin property around the point (x,w).

Then there exists $\varepsilon>0$, such that, for every $(x,y,w,z)\in \operatorname{Gr}(F,G,K)$ around $(\overline{x},\overline{y},\overline{w},\overline{z})$, and for every $\rho\in]0,\varepsilon[$,

$$B_Y(y+w+z,\lambda\rho)\subset (F+G)\left(B_X(x,\rho)\right)+K\left(B_X(x,\rho)\right),$$

and, consequently, $(F+G,K)$ is weakly open at $(\overline{x},\overline{y}+\overline{z}+\overline{w})$.

Remark that if in the previous theorem we take $K(x):=0$ for every $x\in X$, then we obtain Theorem 4.4 from [15], which is an openness result for $F+G$. Furthermore, in case $G(x):=0$ for every $x\in X$, then we obtain the following weak openness result.

Theorem 25 *Let X,Y be Banach spaces, F, $K:X\rightrightarrows Y$ be multifunctions such that $\operatorname{Gr}F$, $\operatorname{Gr}K$ are closed around $(\overline{x},\overline{y})\in\operatorname{Gr}F$ and $(\overline{x},\overline{z})\in\operatorname{Gr}K$, respectively. Suppose, moreover, that there exists $\lambda>0$ such that, for every $(x,y,t,z)\in\operatorname{Gr}F\times\operatorname{Gr}K$ around $(\overline{x},\overline{y},\overline{x},\overline{z})$, the following assumptions are satisfied:*

(i) the next inclusion holds

$$B_Y(0,\lambda)\subset \operatorname{cl}[D_BF(x,y)\left(B_X(0,1)\right)+D_BK(t,z)\left(B_X(0,1)\right)\cap B_Y(0,1)]; \tag{13}$$

(ii) either F is proto-differentiable at x relative to y, or K is proto-differentiable at t relative to z.

Then there exists $\varepsilon>0$, such that, for every $(x,y,z)\in\operatorname{Gr}(F,K)$ around $(\overline{x},\overline{y},\overline{z})$, and for every $\rho\in]0,\varepsilon[$,

$$B_Y(y+z,\lambda\rho)\subset F(B_X(x,\rho))+K\left(B_X(x,\rho)\right), \tag{14}$$

and, consequently, (F,K) is weakly open at $(\overline{x},\overline{y}+\overline{z})$.

Now consider the same type of results, but using the generalized differentiation objects lying in dual spaces. On the dual spaces, we work with the constructions developed by Mordukhovich and his collaborators (see [33]). Some of these concepts are briefly listed here.

Let X be a normed vector space, S be a non-empty subset of X and let $x\in S$, $\varepsilon\geq 0$. The set of $\varepsilon-$normals to S at x is

$$\widehat{N}_\varepsilon(S,x):=\left\{x^*\in X^*\mid \limsup_{u\overset{S}{\to}x}\frac{x^*(u-x)}{\|u-x\|}\leq\varepsilon\right\}, \tag{15}$$

where $u\overset{S}{\to}x$ means that $u\to x$ and $u\in S$.

If $\varepsilon=0$, the elements in the right-hand side of (15) are called Fréchet normals and their collection, denoted by $\widehat{N}(S,x)$, is the Fréchet normal cone to S at x.

Let $\overline{x}\in S$. The basic (or limiting, or Mordukhovich) normal cone to S at $\overline{x}$ is

$$N(S,\overline{x}):=\{x^*\in X^*\mid \exists\varepsilon_n\downarrow 0, x_n\overset{S}{\to}\overline{x}, x_n^*\overset{w^*}{\to}x^*, x_n^*\in\widehat{N}_{\varepsilon_n}(S,x_n),\forall n\in\mathbb{N}\}.$$

If X is an Asplund space, and S is closed around $\overline{x}$, the formula for the basic normal cone looks as follows:

$$N(S,\overline{x})=\{x^*\in X^*\mid \exists x_n\overset{S}{\to}\overline{x}, x_n^*\overset{w^*}{\to}x^*, x_n^*\in\widehat{N}(S,x_n),\forall n\in\mathbb{N}\}. \tag{16}$$

Accordingly, two concepts of coderivatives for set-valued maps are in order.

Let $F:X\rightrightarrows Y$ be a set-valued map and $(\overline{x},\overline{y})\in\operatorname{Gr}F$. Then the Fréchet coderivative of F at $(\overline{x},\overline{y})$ is the set-valued map $\widehat{D}^*F(\overline{x},\overline{y}):Y^*\rightrightarrows X^*$ given by

$$\widehat{D}^*F(\overline{x},\overline{y})(y^*):=\{x^*\in X^*\mid (x^*,-y^*)\in\widehat{N}(\operatorname{Gr}F,(\overline{x},\overline{y}))\}.$$

Similarly, the normal coderivative of F at $(\overline{x},\overline{y})$ is the set-valued map $D^*F(\overline{x},\overline{y}):Y^*\rightrightarrows X^*$ given by

$$D^*F(\overline{x},\overline{y})(y^*):=\{x^*\in X^*\mid (x^*,-y^*)\in N(\operatorname{Gr}F,(\overline{x},\overline{y}))\}.$$

As usual, when $F:=f$ is a function, since $\overline{y}\in F(\overline{x})$ means $\overline{y}=f(\overline{x})$, we write $\widehat{D}^*f(\overline{x})$, and similarly for D^*.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be finite at $\overline{x} \in X$ and lower semicontinuous around $\overline{x}$; the Fréchet subdifferential of f at $\overline{x}$ is the set

$$\widehat{\partial} f(\overline{x}) := \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\},$$

where $\operatorname{epi} f$ denotes the epigraph of f. The basic (or limiting, or Mordukhovich) subdifferential of f at $\overline{x}$ is given by

$$\partial f(\overline{x}) := \{x^* \in X^* \mid (x^*, -1) \in N(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\}.$$

The basic calculus rules for these objects (especially for the subdifferential of the sum) are well-known.

For a cone $K \subset Y$ we denote by

$$K^+ := \{y^* \in Y^* \mid y^*(y) \geq 0, \ \forall y \in K\}$$

its positive dual cone.

Consider the closed subsets $C_1, ..., C_k$ of an Asplund space X. One says that $C_1, ..., C_k$ are allied at $\overline{x} \in C_1 \cap ... \cap C_k$ whenever $(x_{in}) \overset{C_i}{\to} \overline{x}$, $x_{in}^* \in \widehat{N}(C_i, x_{in})$, $i = \overline{1,k}$, the relation $(x_{1n}^* + ... + x_{kn}^*) \to 0$ implies $(x_{in}^*) \to 0$ for every $i = \overline{1,k}$. The concept of alliedness was introduced by Penot in [35] in order to get a calculus rule for the Fréchet normal cone to the intersection of sets. More precisely, if the subsets $C_1, ..., C_k$ are allied at $\overline{x} \in C_1 \cap ... \cap C_k$, then there exists $r > 0$ such that, for every $\varepsilon > 0$ and every $x \in [C_1 \cap ... \cap C_k] \cap B_X(\overline{x}, r)$, there exist $x_i \in C_i \cap B_X(x, \varepsilon)$, $i = \overline{1,k}$ such that

$$\widehat{N}(C_1 \cap ... \cap C_k, x) \subset \widehat{N}(C_1, x_1) + ... + \widehat{N}(C_k, x_k) + \varepsilon D_{X^*}.$$

Before providing the first weak openness result with the assumptions given in terms of generalized differentiation objects on dual spaces, we consider the following sets

$$C_1 := \{(x, y, z) \in X \times Y \times Y \mid y \in F(x)\},$$
$$C_2 := \{(x, y, z) \in X \times Y \times Y \mid z \in G(x)\}.$$

Remark that the alliedness of the sets C_1 and C_2 at $(\overline{x}, \overline{y}, \overline{z}) \in C_1 \cap C_2$ means that for every sequences $(x_n, y_n) \overset{\operatorname{Gr} F}{\to} (\overline{x}, \overline{y})$, $(u_n, v_n) \overset{\operatorname{Gr} G}{\to} (\overline{x}, \overline{z})$, and every $x_n^* \in \widehat{D}^* F(x_n, y_n)(y_n^*)$, $u_n^* \in \widehat{D}^* G(u_n, v_n)(v_n^*)$,

$$(x_n^* + u_n^*) \to 0, \ (y_n^*) \to 0, \ (v_n^*) \to 0 \Rightarrow (x_n^*) \to 0, \ (u_n^*) \to 0.$$

Theorem 26 *Let X, Y be Asplund spaces, $F, G, K : X \rightrightarrows Y$ multifunctions with $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, and $(\overline{x}, 0) \in \operatorname{Gr} G \cap \operatorname{Gr} K$. Assume that the following assumptions are satisfied:*

(i) $\operatorname{Gr} F$, $\operatorname{Gr} G$ and $\operatorname{Gr} K$ are closed around $(\overline{x}, \overline{y})$, $(\overline{x}, 0)$ and $(\overline{x}, 0)$, respectively;

(ii) the sets C_1 and C_2 are allied at $(\overline{x}, \overline{y}, 0)$;

(iii) there exist $c > 0$, $r > 0$ such that for every $(x_1, y_1) \in \operatorname{Gr} F \cap (B_X(\overline{x}, r) \times B_Y(\overline{y}, r))$, $(x_2, y_2) \in \operatorname{Gr} G \cap (B_X(\overline{x}, r) \times B_Y(0, r))$, $(x_3, y_3) \in \operatorname{Gr} K \cap (B_X(\overline{x}, r) \times B_Y(0, r))$, $y^* \in S_{Y^*}$, z_1^*, z_2^*, $z_3^* \in cB_{Y^*}$, $x_1^* \in \widehat{D}^* F(x_1, y_1)(y^* + z_1^*)$, $x_2^* \in \widehat{D}^* G(x_2, y_2)(y^* + z_2^*)$, $x_3^* \in \widehat{D}^* K(x_3, y_3)(y^* + z_3^*)$, we have

$$c \|3y^* + z_1^* + z_2^* + z_3^*\| \leq \|x_1^* + x_2^* + x_3^*\|.$$

Then for every $a \in]0, c[$, there exists $\varepsilon > 0$ such that, for every $\rho \in]0, \varepsilon]$

$$B_Y(\overline{y}, \rho a) \subset (F + G)(B_X(\overline{x}, \rho)) + K(B_X(\overline{x}, \rho)),$$

and, consequently, $(F + G, K)$ is weakly open at $(\overline{x}, \overline{y})$.

Remark that the previous theorem can be formulated for $(\overline{x}, \overline{z}) \in \operatorname{Gr} G$ and $(\overline{x}, \overline{w}) \in \operatorname{Gr} K$ instead of $(\overline{x}, 0) \in \operatorname{Gr} G \cap \operatorname{Gr} K$, but we prefer the present form because of the later use in the paper. Observe that if in Theorem 26 we take $K(x) := 0$ for every $x \in X$, then we obtain Theorem 4.2 from [34], which is an openness result for $F + G$. Furthermore, in case $G(x) := 0$ for every $x \in X$, then we obtain Theorem 5.2 from [16].

Putting together all the facts investigated until now, we are able to formulate necessary optimality conditions for Henig proper nondominated points in VOS setting. The first result, on primal spaces, reads as follows.

Theorem 27 *Let X, Y be Banach spaces, F, G, $K : X \rightrightarrows Y$ be multifunctions such that* $\operatorname{Gr} F$ *and* $\operatorname{Gr} K$ *are closed around* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$ *and* $(\overline{x}, 0) \in \operatorname{Gr} K$*, respectively. Suppose that there exists a neighborhood* U *of* $\overline{x}$ *such that:*

(i) there exists $e \in Y \setminus \{0\}$ *such that* $e \in K(x)$ *for all* $x \in U$*;*

(ii) there exists a constant $L > 0$ *such that for all* $u, v \in U$*,*

$$F(u) \subset F(v) - L \|u - v\| e + K(v);$$

(iii) $K(x)$ *is based for all* $x \in U$ *(the base of* $K(x)$ *is denoted by* $B(x)$*);*

(iv) there exists a constant $l > 0$ *such that for all* $u, v \in U$

$$B(u) \subset B(v) + l \|u - v\| D_Y.$$

Moreover, suppose that for every $(x, y, t, z) \in \operatorname{Gr} F \times \operatorname{Gr} K$ *around* $(\overline{x}, \overline{y}, \overline{x}, 0) \in \operatorname{Gr} F \times \operatorname{Gr} K$*, either* K *is proto-differentiable at* t *relative to* z*, or* F *is proto-differentiable at* x *relative to* y*.*

If $(\overline{x}, \overline{y}) \in \operatorname{VosHEff}(F, M; K)$*, then for every* $\varepsilon > 0$*, there exist*

$$(x_\varepsilon, y_\varepsilon, t_\varepsilon, z_\varepsilon) \in [\operatorname{Gr} F \times \operatorname{Gr} K] \cap [B_X(\overline{x}, \varepsilon) \times B_Y(\overline{y}, \varepsilon) \times B_X(\overline{x}, \varepsilon) \times B_Y(0, \varepsilon)]$$

and $w_\varepsilon \in B_Y(0, \varepsilon) \setminus \{0\}$ *such that*

$$w_\varepsilon \notin \operatorname{cl}[(D_B F(x_\varepsilon, y_\varepsilon) + D_B(\beta d_M(\cdot) e)(x_\varepsilon) \cap B_Y(0,1))(B_X(0,1)) + D_B K(t_\varepsilon, z_\varepsilon)(B_X(0,1)) \cap B_Y(0,1)].$$

In the following we consider as well, besides the sets C_1 and C_2, the set

$$C_3 := \{(x, y, z) \in X \times Y \times Y \mid z \in K(x)\}.$$

We give now our result concerning optimality conditions on dual spaces.

Theorem 28 *Let X be an Asplund space and Y be a finite dimensional space, let F, $K : X \rightrightarrows Y$ be multifunctions such that* $\operatorname{Gr} F$ *and* $\operatorname{Gr} K$ *are closed around* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (M \times Y)$ *and* $(\overline{x}, 0) \in \operatorname{Gr} K$*, respectively. Suppose that there exists a neighborhood* U *of* $\overline{x}$ *such that:*

(i) there exists $e \in \overline{K} \setminus \{0\}$*, where* $\overline{K} := \bigcap_{x \in U} K(x)$*;*

(ii) there exists a constant $L > 0$ *such that for all* $u, v \in U$*,*

$$F(u) \subset F(v) - L \|u - v\| e + K(v);$$

(iii) $K(x)$ *is based for all* $x \in U$ *(the base of* $K(x)$ *is denoted by* $B(x)$*);*

(iv) there exists a constant $l > 0$ *such that for all* $u, v \in U$

$$B(u) \subset B(v) + l \|u - v\| D_Y.$$

Moreover, suppose that the sets C_1 *and* C_3 *are allied at* $(\overline{x}, \overline{y}, 0)$*.*

If $(\overline{x}, \overline{y}) \in \operatorname{VosHEff}(F, M; K)$*, then for all* $\beta > L$*, there exists* $y^* \in \overline{K}^+ \setminus \{0\}$ *such that*

$$0 \in D^* F(\overline{x}, \overline{y})(y^*) + D^* K(\overline{x}, 0)(y^*) + \beta y^*(e) \partial d_M(\overline{x}). \tag{17}$$

The book chapter [9] surveys and improves the theory of vector optimization with variable ordering structures for the case when both the objective and ordering mappings vary with the same argument. On one hand, we aim at a complete comparison of the efficiencies concepts introduced in this setting, and, on the other hand, we present some enhanced necessary optimality conditions based on a penalization method involving a special instance of the minimal time function and on two openness results, in primal and dual spaces, for a family of set-valued mappings.

The paper [17] studies some constrained vector optimization problems on an approach in dual spaces, working as well with the generalized differentiation constructions developed by Mordukhovich. More precisely, we study the concept of nondomination, which is also known in the literature as Pareto minimum (or, Pareto efficiency), in constrained multiobjective optimization with respect to variable ordering structures, from the viewpoint of necessary optimality conditions.

We present the vector optimization problem considered, and we define the concept of solution associated to the proposed problem and we give some notations for the generalized differentiation objects used here.

Let m be a nonzero natural number and $F_i : X \rightrightarrows Y_i$ be closed set-valued maps for every $i \in \overline{0,m}$, where X and each Y_i are Asplund spaces. Let $K_0 : X \rightrightarrows Y_0$ be a closed set-valued map such that $K_0(x)$ is a convex, proper and pointed cone in Y_0 for any $x \in X$. Then, for every $x \in X$, the cone $K_0(x)$ introduces an order relation on Y_0 by the equivalence:

$$y_1 \leq_{K_0(x)} y_2 \Leftrightarrow y_2 - y_1 \in K_0(x).$$

For every $i \in \overline{1,m}$ we consider the closed set-valued maps $K_i : X \rightrightarrows Y_i$ such that $K_i(x)$ is a convex and proper cone (not necessarily pointed) in Y_i for any $x \in X$.

We consider the following vector optimization problem

$$\min\nolimits_{K_0} F_0(x), \text{ such that } F_i(x) \cap (-K_i(x)) \neq \emptyset,\ i \in \overline{1,m},\ x \in \Omega, \tag{18}$$

where Ω is a closed and nonempty subset of X.

Take $(\overline{x}, \overline{y}_0, ..., \overline{y}_m) \in \Omega \times \prod_{i=0}^{m} Y_i$ such that $(\overline{x}, \overline{y}_0) \in \operatorname{Gr} F_0$ and, for every $i \in \overline{1,m}$, $\overline{y}_i \in F_i(\overline{x}) \cap -K_i(\overline{x})$. Let $S \subset X$ denote the feasible set of (18), that is,

$$S := \left\{x \in \Omega \colon 0 \in (F_i + K_i)(x),\ \forall i \in \overline{1,m}\right\}.$$

The point $(\overline{x}, \overline{y}_0)$ is a local solution of (18) if there exists $U \in \mathcal{V}(\overline{x})$ such that for every $x \in S \cap U$ one has

$$(F_0(x) - \overline{y}_0) \cap (-K_0(x)) \subset \{0\}$$

(for more details on this notion, see [15]).

We reconsider the main ideas developed in [44] for the case of a functional constrained vector optimization problem with fixed order structure, to the case of variable order structures. In this demarche, we have to solve several technical questions that arise in the latter, more general case. In particular, the investigation we propose has to take into account the presence of the set-valued map that defines the order with respect to which the efficiency concept we work with is designed, namely K_0, and also the set-valued maps that appear in the definition of the constraints, i.e., K_i with $i \in \overline{1,m}$.

The main tool used in order to achieve our main goal is the following version of an extended extremal principal (for more details see [44, Lemma 2.1]).

Lemma 29 *Let Y be an Asplund space and $A_1, A_2, ..., A_n \subset Y$ be closed sets such that $\bigcap_{i=1}^{n} A_i = \emptyset$. Let $a_i \in A_i$ (for $i \in \overline{1,n}$) and $\varepsilon > 0$ be such that*

$$\sum_{i=1}^{n-1} \|a_i - a_n\| < \gamma(A_1, A_2, ..., A_n) + \varepsilon,$$

where

$$\gamma(A_1, A_2, ..., A_n) := \inf\left\{\sum_{i=1}^{n-1} \|a_i - a_n\| :\ (a_1, a_2, ..., a_n) \in A_1 \times A_2 \times ... \times A_n\right\}.$$

Then for every $\lambda > 0$, there exist $\widetilde{a}_i \in A_i$ and $a_i^* \in Y^*$ such that

$$\sum_{i=1}^{n} \|a_i - \widetilde{a}_i\| < \lambda,\ a_i^* \in \widehat{N}(A_i, \widetilde{a}_i) + \frac{\varepsilon}{\lambda} D_{Y^*},$$

$$\sum_{i=1}^{n} \|a_i^*\| = 1 \text{ and } \sum_{i=1}^{n} a_i^* = 0.$$

In the above notations, applying Lemma 29 for the constants $\varepsilon = \frac{1}{k^2}$ and $\lambda = \frac{1}{k}$, with $k \in \mathbb{N}^*$, the points $a_i := (\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_m)$, for every $i \in \overline{0, 2m+1}$, $a_{2m+2} := (\overline{x}, \overline{y}_0 - s_k c_0, \overline{y}_1, ..., \overline{y}_m)$ and the sets

$$A_i := \left\{ (x, y_0, y_1, ..., y_m) \in X \times \prod_{i=0}^{m} Y_i \colon y_i \in F_i(x) \right\}, \ \forall i \in \overline{0, m},$$
$$A_{m+i} := \left\{ (x, y_0, y_1, ..., y_m) \in X \times \prod_{i=0}^{m} Y_i \colon y_i \in -K_i(x) \right\}, \ \forall i \in \overline{1, m},$$
$$A_{2m+1} := \left\{ (x, y_0, y_1, ..., y_m) \in X \times \prod_{i=0}^{m} Y_i \colon x \in \Omega \cap D(\overline{x}, \delta) \right\},$$
$$A_{2m+2} := \left\{ (x, y_0, y_1, ..., y_m) \in X \times \prod_{i=0}^{m} Y_i \colon y_0 \in \overline{y}_0 - s_k c_0 - K_0(x) \right\}.$$

we are able to obtain, by using the Fréchet coderivatives of the set-valued maps F_i and K_i for every $i \in \overline{0, m}$ and the Fréchet normal cone to Ω, the following result concerning some necessary optimality conditions for a solution of problem (18).

Theorem 30 *Let $(\overline{x}, \overline{y}_0) \in \operatorname{Gr} F_0 \cap (\Omega \times Y_0)$ be a local solution for problem (18) and the points $\overline{y}_i \in F_i(\overline{x}) \cap -K_i(\overline{x})$ for every $i \in \overline{1, m}$. Suppose that there exists $W \in \mathcal{V}(\overline{x})$ such that $\bigcap_{x \in \Omega \cap W} K_0(x) \neq \{0\}$. Then one of the following assertions holds:*

(i) for every $\varepsilon > 0$, $i \in \overline{0, m}$, and $j \in \overline{1, m}$, there exist $x_i, \widetilde{x}_0, \widetilde{x}_j \in \overline{x} + \varepsilon D_X$, $y_i \in (\overline{y}_i + \varepsilon D_{Y_i}) \cap F_i(x_i)$, $\widetilde{y}_0 \in (\varepsilon D_{Y_0}) \cap K_0(\widetilde{x}_0)$, $\widetilde{y}_j \in \left(-\overline{y}_j + \varepsilon D_{Y_j}\right) \cap K_j(\widetilde{x}_j)$, $\omega \in \Omega \cap (\overline{x} + \varepsilon D_X)$, $y_i^* \in Y_i^*$ such that $\sum_{i=0}^{m} \|y_i^*\| = 1$ and

$$0 \in \sum_{i=0}^{m} \left(\widehat{D}^* F_i(x_i, y_i) \left(y_i^* + \varepsilon D_{Y_i^*}\right) \cap M B_{X^*} \right) + \sum_{i=0}^{m} \left(\widehat{D}^* K_i(\widetilde{x}_i, \widetilde{y}_i) \left(y_i^* + \varepsilon D_{Y_i^*}\right) \cap M B_{X^*} \right)$$
$$+ \widehat{N}(\Omega, \omega) \cap M B_{X^*} + \varepsilon D_{X^*},$$

where $M > 0$ is a constant independent of ε;

(ii) for every $\varepsilon > 0$, $i \in \overline{0, m}$, and $j \in \overline{1, m}$, there exist $x_i, \widetilde{x}_0, \widetilde{x}_j \in \overline{x} + \varepsilon D_X$, $y_i \in (\overline{y}_i + \varepsilon D_{Y_i}) \cap F_i(x_i)$, $\widetilde{y}_0 \in (\varepsilon D_{Y_0}) \cap K_0(\widetilde{x}_0)$, $\widetilde{y}_j \in \left(-\overline{y}_j + \varepsilon D_{Y_j}\right) \cap K_j(\widetilde{x}_j)$, $\omega \in \Omega \cap (\overline{x} + \varepsilon D_X)$, $x_i^* \in \widehat{D}^* F_i(x_i, y_i)(\varepsilon D_{Y_i^*})$, $\widetilde{x}_i^* \in \widehat{D}^* K_i(\widetilde{x}_i, \widetilde{y}_i) \left(\varepsilon D_{Y_i^*}\right)$, $\omega^* \in \widehat{N}(\Omega, \omega) + \varepsilon D_{X^*}$ such that

$$\|\omega^*\| + \sum_{i=0}^{m} (\|x_i^*\| + \|\widetilde{x}_i^*\|) = 1 \text{ and } \omega^* + \sum_{i=0}^{m} (x_i^* + \widetilde{x}_i^*) = 0.$$

Further, we proved that the conclusion (ii) of Theorem 30 does not hold, when we impose some alliedness hypothesis and $\Omega = X$.

Corollary 31 *Let $(\overline{x}, \overline{y}_0) \in \operatorname{Gr} F_0$ be a local solution for problem (18) and the points $\overline{y}_i \in F_i(\overline{x}) \cap -K_i(\overline{x})$ for every $i \in \overline{1, m}$. Suppose that there exists W a neighborhood of $\overline{x}$ such that $\bigcap_{x \in W} K_0(x) \neq \{0\}$. If the closed sets $C_0, C_1, ..., C_{2m+1}$ are allied at $(\overline{x}, \overline{y}_0, ..., \overline{y}_m, 0, -\overline{y}_1, -\overline{y}_2, ..., -\overline{y}_m) \in \bigcap_{i=0}^{2m+1} C_i$, where for every $i \in \overline{0, m}$,*

$$C_i := \left\{ (x, y_0, y_1, ..., y_{2m+1}) \in X \times Y^{2m+2} \colon y_i \in F_i(x) \right\}, \tag{19}$$
$$C_{m+i+1} := \left\{ (x, y_0, y_1, ..., y_{2m+1}) \in X \times Y^{2m+2} \colon y_{m+i+1} \in K_i(x) \right\},$$

then for every $\varepsilon > 0$, $i \in \overline{0, m}$, and $j \in \overline{1, m}$, there exist $x_i, \widetilde{x}_0, \widetilde{x}_j \in \overline{x} + \varepsilon D_X$, $y_i \in (\overline{y}_i + \varepsilon D_{Y_i}) \cap F_i(x_i)$, $\widetilde{y}_0 \in (\varepsilon D_{Y_0}) \cap K_0(\widetilde{x}_0)$, $\widetilde{y}_j \in \left(-\overline{y}_j + \varepsilon D_{Y_j}\right) \cap K_j(\widetilde{x}_j)$, $\omega \in \Omega \cap (\overline{x} + \varepsilon D_X)$, $y_i^* \in Y_i^*$ such that $\sum_{i=0}^{m} \|y_i^*\| = 1$ and

$$0 \in \sum_{i=0}^{m} \left(\widehat{D}^* F_i(x_i, y_i) \left(y_i^* + \varepsilon D_{Y_i^*}\right) \cap M B_{X^*} \right) + \sum_{i=0}^{m} \left(\widehat{D}^* K_i(\widetilde{x}_i, \widetilde{y}_i) \left(y_i^* + \varepsilon D_{Y_i^*}\right) \cap M B_{X^*} \right) + \varepsilon D_{X^*}, \tag{20}$$

where $M > 0$ is a constant independent of ε.

In the following, we tackle again the general case and we present some necessary optimality conditions for a local solution of problem (18) using Mordukhovich coderivatives and the basic normal cone. Mention that for this passing from the approximate optimality conditions presented in Theorem 30, to exact optimality conditions we have to impose some specific conditions of sequential normal compactness (for more details, see [33, pages 27, 76, 266]).

Theorem 32 *Let $(\overline{x},\overline{y}_0) \in \operatorname{Gr} F_0 \cap (\Omega \times Y_0)$ be a local solution for problem (18) and the points $\overline{y}_i \in F_i(\overline{x}) \cap -K_i(\overline{x})$ for every $i \in \overline{1,m}$. Suppose that there exists $W \in \mathcal{V}(\overline{x})$ such that $\overline{K}_i := \bigcap_{x\in\Omega\cap W} K_i(x) \neq \{0\}$ for every $i \in \overline{0,m}$. If for every $i \in \overline{0,m}$, $j \in \overline{1,m}$ the sets $\overline{K}_i$ are sequentially normally compact in 0 and the set-valued maps F_i, K_0 and K_j are partially sequentially normally compact in $(\overline{x},\overline{y}_i) \in \operatorname{Gr} F_i$, $(\overline{x},0) \in \operatorname{Gr} K_0$, respectively $(\overline{x},-\overline{y}_j) \in \operatorname{Gr} K_j$, then one of the following assertions holds:*

(i) for every $i \in \overline{0,m}$, there exist $y_i^* \in \overline{K}_i^+$ such that $\sum_{i=0}^{m} \|y_i^*\| = 1$ and

$$0 \in \sum_{i=1}^{m} \left[D^*F_i(\overline{x},\overline{y}_i)(y_i^*) + D^*K_i(\overline{x},-\overline{y}_i)(y_i^*)\right] + D^*F_0(\overline{x},\overline{y}_0)(y_0^*) + D^*K_0(\overline{x},0)(y_0^*) + N(\Omega,\overline{x}),$$

where $\overline{K}_i^+$ denotes the positive dual cone of K_i;

(ii) for every $i \in \overline{0,m}$, and $j \in \overline{1,m}$, there exist $x_i^* \in D^*F_i(\overline{x},\overline{y}_i)(0)$, $\widetilde{x}_0^* \in D^*K_0(\overline{x},0)(0)$, $\widetilde{x}_j^* \in D^*K_j(\overline{x},-\overline{y}_j)(0)$, $\omega^* \in N(\Omega,\overline{x})$ such that

$$\omega^* + \sum_{i=0}^{m} (x_i^* + \widetilde{x}_i^*) = 0 \text{ and } \|\omega^*\| + \sum_{i=0}^{m} (\|x_i^*\| + \|\widetilde{x}_i^*\|) = 1. \tag{21}$$

In order to illustrate Theorem 32, we consider the following examples.

Similar to Corollary 31, under some alliedness conditions and for $\Omega = X$, one gets the following result.

Corollary 33 *Let $(\overline{x},\overline{y}_0) \in \operatorname{Gr} F_0$ be a local solution for problem (18) and the points $\overline{y}_i \in F_i(\overline{x}) \cap -K_i(\overline{x})$ for every $i \in \overline{1,m}$. Suppose that there exists W a neighborhood of $\overline{x}$ such that $\overline{K}_i := \bigcap_{x\in W} K_i(x) \neq \{0\}$ for every $i \in \overline{0,m}$. If for every $i \in \overline{0,m}$ the sets $\overline{K}_i$ are sequentially normally compact in 0 and the closed sets $C_0, C_1, ..., C_{2m+1}$ are allied at $(\overline{x},\overline{y}_0,...,\overline{y}_m,0,-\overline{y}_1,-\overline{y}_2,...,-\overline{y}_m) \in \bigcap_{i=0}^{2m+1} C_i$, where the sets $C_0, C_1, ..., C_{2m+1}$ are given in (19), then for every $i \in \overline{0,m}$, there exist $y_i^* \in \overline{K}_i^+$ such that $\sum_{i=0}^{m} \|y_i^*\| = 1$ and*

$$0 \in \sum_{i=1}^{m} \left[D^*F_i(\overline{x},\overline{y}_i)(y_i^*) + D^*K_i(\overline{x},-\overline{y}_i)(y_i^*)\right] + D^*F_0(\overline{x},\overline{y}_0)(y_0^*) + D^*K_0(\overline{x},0)(y_0^*).$$

By simply taking the particular cases our approach cover, we are able to reobtain several results from the literature, in their original forms or in some variations.

In the paper [4] we select two tools of investigation of the classical metric regularity of set-valued mappings, namely the Ioffe criterion and the Ekeland Variational Principle, which we adapt to the study of the directional setting. In this way, we obtain in a unitary manner new and generalized results concerning sufficient conditions for directional metric regularity of a mapping, with applications to the stability of this property at composition and sum of set-valued maps. In this process, we introduce as well new directional tangent cones and the associated generalized differentiation objects and concepts on primal spaces. Moreover, we underline several links between our main assertions by providing alternative proofs for several results.

Let X be a normed vector space, $\emptyset \neq \Omega \subset X$ and $\emptyset \neq L \subset S_X$. Then the function

$$X \ni x \longmapsto T_L(x,\Omega) := \inf\{t \geq 0 \mid \exists \ell \in L : x + t\ell \in \Omega\} \tag{22}$$

is called the directional minimal time function with respect to L. Many properties of this function were systematically analyzed in [13]. Remark that

$$T_L(x,\Omega) < +\infty \text{ if and only if } x \in \Omega - \operatorname{cone} L$$

and

$$d(x,\Omega) \le T_L(x,\Omega) \text{ for all } x \in X.$$

Moreover, if $L = S_X$, then $T_L(\cdot,\Omega) = d(\cdot,\Omega)$. We shall sometimes use the notation T_L instead of $T_L(\cdot,\Omega)$, when no danger of confusion arises. Moreover, if $\Omega = \{u\}$ for a point $u \in X$, we denote in what follows $T_L(\cdot,\{u\})$ by $T_L(\cdot,u)$. Clearly, for each x, $u \in X$, if $T_L(x,u) < +\infty$ (which is equivalent to $u - x \in \operatorname{cone} L$), then

$$T_{-L}(u,x) = T_L(x,u) = \|u - x\|.$$

Definition 34 *Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ with $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ and sets $L \subset S_X$ and $M \subset S_Y$ be nonempty.*

(i) One says that F is directionally metrically regular around $(\overline{x},\overline{y})$ with respect to L and M with a constant $c > 0$ *if there are $\varepsilon > 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that, for every $(x,y) \in U \times V$ such that $T_M(y,F(x)) < \varepsilon$,*

$$T_L(x,F^{-1}(y)) \le c \cdot T_M(y,F(x)). \tag{23}$$

The modulus of directional regularity of F around $(\overline{x},\overline{y})$ with respect to L and M, *denoted by* $\operatorname{dirreg}_{L\times M} F(\overline{x},\overline{y})$, *is the infimum of $c > 0$ such that F is directionally metrically regular around $(\overline{x},\overline{y})$ with respect to L and M with the constant c.*

(ii) One says that F is directionally linearly open around $(\overline{x},\overline{y})$ with respect to L and M with a constant $c > 0$ *if there are $\varepsilon > 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that, for every $r \in (0,\varepsilon)$ and every $(x,y) \in (U \times V) \cap \operatorname{Gr} F$,*

$$B(y,cr) \cap (y - \operatorname{cone} M) \subset F(B(x,r) \cap (x + \operatorname{cone} L)). \tag{24}$$

The modulus of directional openness of F around $(\overline{x},\overline{y})$ with respect to L and M, *denoted by* $\operatorname{dirsur}_{L\times M} F(\overline{x},\overline{y})$, *is the supremum of $c > 0$ such that F is directionally linearly open around $(\overline{x},\overline{y})$ with respect to L and M with the constant c.*

(iii) One says that F has the directional Aubin property around $(\overline{x},\overline{y})$ with respect to L and M with a constant $c > 0$ *if there are neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that, for every $x, u \in U$,*

$$e_M(F(x) \cap V, F(u)) \le c \cdot T_L(u,x). \tag{25}$$

The modulus of the directional Aubin property of F around $(\overline{x},\overline{y})$ with respect to L and M, *denoted by* $\operatorname{dirlip}_{L\times M} F(\overline{x},\overline{y})$, *is the infimum of $c > 0$ such that F has the directional Aubin property around $(\overline{x},\overline{y})$ with respect to L and M with the constant c.*

Of course, when $L := S_X$ and $M := S_Y$, the previous concepts reduce to the usual metric regularity, linear openness, and Aubin property around the reference point (see, e.g., [6] for more details).

The next result contains the announced link between the notions given before (see [14, Proposition 2.3]). The convention $1/0 = +\infty$ applies here.

Proposition 35 *Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ with $(\overline{x},\overline{y}) \in \operatorname{Gr} F$and sets $L \subset S_X$ and $M \subset S_Y$ be nonempty. Then*

$$\operatorname{dirreg}_{L\times M} F(\overline{x},\overline{y}) = (\operatorname{dirsur}_{L\times M} F(\overline{x},\overline{y}))^{-1} = \operatorname{dirlip}_{M\times L} F^{-1}(\overline{y},\overline{x}).$$

Moreover, a direct comparison of these concept with other directional concepts of regularity is given.

The first Ioffe type criterion is given next for single-valued maps

Proposition 36 *Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ be Banach spaces. Consider a nonempty closed subset L of S_X such that* $\operatorname{cone} L$ *is convex, a nonempty closed subset M of S_Y, a point $\overline{x} \in X$, and a mapping $g : X \to Y$ such that there is a neighborhood U of $\bar{x}$ such that the set $D := U \cap \operatorname{Dom} g$ is closed and g is continuous on D. Then* $\operatorname{dirsur}_{L\times M} g(\overline{x})$ *equals to the supremum of $c > 0$ for which there is $r > 0$ such that for all $(x,y) \in (B[\overline{x},r] \cap \operatorname{Dom} g) \times B[g(\overline{x}),r]$, with $0 < T_M(y,g(x)) < +\infty$, there is a point $x' \in \operatorname{Dom} g$ satisfying*

$$cT_L(x,x') < T_M(y,g(x)) - T_M(y,g(x')).$$

For set-valued maps we derive the following assertion.

Proposition 37 *Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ be Banach spaces. Consider nonempty closed subsets L of S_X and M of S_Y such that* $\operatorname{cone} L$ *and* $\operatorname{cone} M$ *are convex, a point $(\overline{x},\overline{y}) \in X\times Y$, and a set-valued mapping $F : X \rightrightarrows Y$ the graph of which is locally closed near $(\overline{x},\overline{y}) \in \operatorname{Gr} F$. Then $\operatorname{dirsur}_{L\times M} F(\overline{x},\overline{y})$ equals to the supremum of all $c>0$ for which there are $r>0$ and $\alpha \in (0,1/c)$ such that for any $(x,v) \in (B[\overline{x},r]\times B[\overline{y},r]) \cap \operatorname{Gr} F$ and any $y \in B[\overline{y},r]$, with $0 < T_M(y,v) < +\infty$, there is a pair $(x',v') \in \operatorname{Gr} F$ such that*

$$c \max\{T_L(x,x'), \alpha\|v-v'\|\} < T_M(y,v) - T_M(y,v'). \tag{26}$$

In what follows, we speak about the local stability at composition of a pair multifunctions, which essentially says that a point from the graph of the composed multifunction, close to the reference one, can be written by the use of points from the graphs of the involved set-valued maps, which are also close to the corresponding reference ones. Given metric spaces (X,ϱ), (Y,ϱ), and (Z,ϱ), a composition of set-valued mappings $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ is the mapping $G\circ F : X \rightrightarrows Z$ defined by

$$(G\circ F)(x) := \bigcup_{y\in F(x)} G(y), \quad x\in X;$$

and a product of set-valued mappings $F_1 : X \rightrightarrows Y$ and $F_2 : X \rightrightarrows Z$ is the mapping $(F_1,F_2) : X \rightrightarrows Y\times Z$ defined by

$$(F_1,F_2)(x) := F_1(x)\times F_2(x), \quad x\in X.$$

Definition 38 *Let (X,ϱ), (Y,ϱ), and (Z,ϱ) be metric spaces and $(\overline{x},\overline{y},\overline{z}) \in X\times Y\times Z$ be fixed. Consider set-valued mappings $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ such that $\overline{y} \in F(\overline{x})$ and $\overline{z} \in G(\overline{y})$. We say that the pair F, G is* composition-stable around $(\overline{x},\overline{y},\overline{z})$ *if for every $\varepsilon>0$ there exists $\delta>0$ such that, for every $x\in B(\overline{x},\delta)$ and every $z \in (G\circ F)(x)\cap B(\overline{z},\delta)$, there exists $y \in F(x)\cap B(\overline{y},\varepsilon)$ such that $z\in G(y)$.*

We present next one of the main results, which asserts the stability of directional regularity under composition.

Theorem 39 *Let $(X,\|\cdot\|)$, $(Y,\|\cdot\|)$, $(Z,\|\cdot\|)$, and $(W,\|\cdot\|)$ be Banach spaces and $(\overline{x},\overline{y},\overline{z},\overline{w}) \in X\times Y\times Z\times W$ be fixed. Consider nonempty closed subsets L of S_X, M of S_Y, N of S_Z, and P of S_W such that* $\operatorname{cone} L$, $\operatorname{cone} M$, $\operatorname{cone} N$, *and* $\operatorname{cone} P$ *are convex, set-valued mappings $F_1 : X \rightrightarrows Y$, $F_2 : X \rightrightarrows Z$, and $G : Y\times Z \rightrightarrows W$ such that F_1 has a locally closed graph near $(\overline{x},\overline{y}) \in \operatorname{Gr} F_1$, F_2 has a locally closed graph near $(\overline{x},\overline{z}) \in \operatorname{Gr} F_2$, and G has a locally closed graph near $(\overline{y},\overline{z},\overline{w}) \in \operatorname{Gr} G$. Define the mapping $\mathcal{E}_{G,(F_1,F_2)} : X\times Y\times Z \rightrightarrows W$ by*

$$\mathcal{E}_{G,(F_1,F_2)}(x,y,z) := \begin{cases} G(y,z), & \text{if } (y,z)\in (F_1,F_2)(x),\\ \emptyset, & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}\operatorname{dirsur}_{L\times M\times N\times P} \mathcal{E}_{G,(F_1,F_2)}(\overline{x},\overline{y},\overline{z},\overline{w}) \geq{}& \operatorname{dirsur}_{L\times M} F_1(\overline{x},\overline{y}) \cdot \widehat{\operatorname{dirsur}}^{\,y}_{-M\times P} G(\overline{y},\overline{z},\overline{w})\\ &- \operatorname{dirlip}_{-L\times N} F_2(\overline{x},\overline{z}) \cdot \widehat{\operatorname{dirlip}}^{\,z}_{-N\times P} G(\overline{y},\overline{z},\overline{w}).\end{aligned} \tag{27}$$

If, in addition, the pair $(F_1,F_2), G$ is composition-stable around $(\overline{x},(\overline{y},\overline{z}),\overline{w})$, then

$$\begin{aligned}\operatorname{dirsur}_{L\times P} (G\circ(F_1,F_2))(\overline{x},\overline{w}) \geq{}& \operatorname{dirsur}_{L\times M} F_1(\overline{x},\overline{y}) \cdot \widehat{\operatorname{dirsur}}^{\,y}_{-M\times P} G(\overline{y},\overline{z},\overline{w})\\ &- \operatorname{dirlip}_{-L\times N} F_2(\overline{x},\overline{z}) \cdot \widehat{\operatorname{dirlip}}^{\,z}_{-N\times P} G(\overline{y},\overline{z},\overline{w}).\end{aligned} \tag{28}$$

Moreover, we employ a directional version of the Bouligand (graphical) derivative to derive sufficient conditions for the directional regularity.

Definition 40 *Let Ω be a nonempty subset of a normed space $(X,\|\cdot\|)$, $M\subset S_X$ be a nonempty set, and $\overline{x}\in\Omega$. The* Bouligand-Severi tangent cone to Ω at $\overline{x}$ with respect to M *is the set*

$$T(\Omega,\overline{x},M) = \left\{u\in X \mid \liminf_{t\downarrow 0}\ t^{-1} T_M(\overline{x}+tu,\Omega) = 0\right\}. \tag{29}$$

Similar to the classical case, one can introduce the directional Ursescu cone, as follows.

Definition 41 *Let Ω be a nonempty subset of a normed space $(X, \|\cdot\|)$, $M \subset S_X$ be a nonempty set, and $\overline{x} \in \Omega$. The* adjacent cone to Ω at $\overline{x}$ with respect to M *is the set*

$$T^\flat(\Omega, \overline{x}, M) = \left\{ u \in X \mid \lim_{t \downarrow 0} \; t^{-1} T_M(\overline{x} + tu, \Omega) = 0 \right\}, \tag{30}$$

that is,

$$T^\flat(\Omega, \overline{x}, M) = \{u \in X \mid \forall (t_n) \downarrow 0, \; \exists (u_n) \subset u + \operatorname{cone} M, \; u_n \to u, \; \forall n \in \mathbb{N}, \; \overline{x} + t_n u_n \in \Omega\}.$$

It is clear that, in general,

$$T^\flat(\Omega, \overline{x}, M) \subset T(\Omega, \overline{x}, M).$$

Definition 42 *Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ with $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, $L \subset S_X$ and $M \subset S_Y$ be nonempty sets.*

(i) The Bouligand derivative of F at $(\overline{x}, \overline{y})$ with respect to L and M *is the set-valued mapping $D_{L,M}F(\overline{x}, \overline{y})$ from X into Y defined by*

$$\begin{aligned} D_{L,M}F(\overline{x}, \overline{y})(u) = &\{v \in Y \mid \exists (t_n) \downarrow 0, \; \exists (u_n) \subset u + \operatorname{cone} L, \; u_n \to u, \; \exists (v_n) \subset v + \operatorname{cone} M, \; (v_n) \to v, \\ &\forall n \in \mathbb{N}, \; \overline{y} + t_n v_n \in F(\overline{x} + t_n u_n)\}. \end{aligned}$$

(ii) The adjacent derivative of F at $(\overline{x}, \overline{y})$ with respect to L and M *is the set-valued mapping denoted $D^\flat_{L,M}F(\overline{x}, \overline{y})$ from X into Y defined similarly.*

(iii) One says that F is directionally proto-differentiable with respect to $L \times M$ at $\overline{x}$ relative to $\overline{y}$ if $D_{L,M}F(\overline{x}, \overline{y}) = D^\flat_{L,M}F(\overline{x}, \overline{y})$.

Observe that if $\operatorname{cone} \widetilde{L} = \operatorname{cone} L \times \operatorname{cone} M$, with an appropriate choice of $\widetilde{L} \subset S_{X \times Y}$, then

$$\begin{aligned} &\operatorname{Gr} D_{L,M}F(\overline{x}, \overline{y}) = T(\operatorname{Gr} F, (\overline{x}, \overline{y}), \widetilde{L}) \text{ and} \\ &\operatorname{Gr} D^\flat_{L,M}F(\overline{x}, \overline{y}) = T^\flat(\operatorname{Gr} F, (\overline{x}, \overline{y}), \widetilde{L}). \end{aligned}$$

One has the following statement.

Theorem 43 *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces. Consider nonempty closed subsets L of S_X and M of S_Y such that $\operatorname{cone} L$ and $\operatorname{cone} M$ are convex and a mapping $F : X \rightrightarrows Y$ the graph of which is locally closed near $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Assume that there are positive constants β, ϱ, and r such that for every $(x, v) \in (B[\overline{x}, r] \times B[\overline{y}, r]) \cap \operatorname{Gr} F$ we have*

$$D_{L,M}F(x, v)(\mathbb{B}_X \cap \operatorname{cone} L) + B[0, \beta] \cap (-\operatorname{cone} M) \supset -(\beta + \varrho)M.$$

Then $\operatorname{dirsur}_{L \times M} F(\overline{x}, \overline{y}) \geq \varrho$.

Finally, we formulate results that use Theorem 43 in order to give primal sufficient conditions for the directional metric regularity of compositions and sums. Note that the next theorem is new even for the non-directional case. For the next results, in the notation of Definition 23, we denote by $D_{L,M}F(x, y) \cap \mathbb{B}_Y \cap \operatorname{cone} M$ the multifunction $H : X \rightrightarrows Y$ given by

$$H(a) = D_{L,M}F_1(x, y)(a) \cap \mathbb{B}_Y \cap \operatorname{cone} M.$$

Theorem 44 *Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$, $(Z, \|\cdot\|)$, and $(W, \|\cdot\|)$ be Banach spaces and $(\overline{x}, \overline{y}, \overline{z}, \overline{w}) \in X \times Y \times Z \times W$ be fixed. Consider the nonempty closed subsets L of S_X, M of S_Y, N of S_Z, and P of S_W such that* $\operatorname{cone} L$, $\operatorname{cone} M$, $\operatorname{cone} N$, *and* $\operatorname{cone} P$ *are convex, the set-valued mappings $F_1 : X \rightrightarrows Y$, $F_2 : X \rightrightarrows Z$, and $G : Y \times Z \rightrightarrows W$ such that F_1 has a locally closed graph near $(\overline{x}, \overline{y}) \in \operatorname{Gr} F_1$, F_2 has a locally closed graph near $(\overline{x}, \overline{z}) \in \operatorname{Gr} F_2$, and G has a locally closed graph near $(\overline{y}, \overline{z}, \overline{w}) \in \operatorname{Gr} G$. Suppose, moreover, that there exist the positive constants β, ϱ, and r such that for every $(x, y, z, w) \in (B[\overline{x}, r] \times B[\overline{y}, r] \times B[\overline{z}, r] \times B[\overline{w}, r]) \cap \operatorname{Gr} \mathcal{E}_{G,(F_1,F_2)}$ we have:*

(i) the next relation holds

$$D_{M,N,P}G(y,z,w)(D_{L,M}F_1(x,y)\cap \mathbb{B}_Y\cap \operatorname{cone} M, D_{L,N}F_2(x,z)\cap \mathbb{B}_Z\cap \operatorname{cone} N)(\mathbb{B}_X\cap \operatorname{cone} L) + B[0,\beta]\cap(-\operatorname{cone} P)\supset -(\beta+\varrho)P;$$

(ii) either F_1 is directionally proto-differentiable with respect to $L\times M$ at x relative to y or F_2 is directionally proto-differentiable with respect to $L\times N$ at x relative to z;

(iii) either F_1 has the directional Aubin property with respect to S_X and M around (x,y) or F_2 has the directional Aubin property with respect to S_X and N around (x,z);

(iv) G is directionally proto-differentiable with respect to $M\times N\times P$ at (y,z) relative to w;

(v) G has the directional Aubin property with respect to $S_Y\times S_Z$ and P around (y,z,w);

(vi) the pair $(F_1,F_2),G$ is composition-stable around $(x,(y,z),w)$.

Then $\operatorname{dirsur}_{L\times P}[G\circ(F_1,F_2)](\overline{x},\overline{w})\geq\varrho$.

The paper ends with a full description of the implications between the main analytical tools we use.

In the paper [12] we study two main situations when the limit of Pareto minima of a sequence of perturbations of a set-valued map F is a critical point of F. The concept of criticality is understood in the Fermat generalized sense by means of limiting (Mordukhovich) coderivative. Firstly, we consider perturbations of enlargement type which, in particular, cover the case of perturbation with dilating cones. Secondly, we study the case of Aubin type perturbations, and for this we introduce and study a new concept of openness with respect to a cone.

Let $K\subset Y$ be a convex closed cone and, additionally, we suppose it is as well pointed (that is, $K\cap -K=\{0\}$) and proper (that is, $K\neq\{0\}$). Consider $F:X\rightrightarrows Y$ as a set-valued mapping, and introduce the unconstrained optimization problem where F is the objective

$$(P)\quad \text{minimize } F(x), \text{ subject to } x\in X.$$

The standard Pareto minimality for this problem is stated in the next definition as the efficiency with respect to the partial order $\leq_K$ induced on Y by K on the basis of the equivalence $y_1\leq_K y_2$ iff $y_2-y_1\in K$.

Definition 45 *A point $(\overline{x},\overline{y})\in\operatorname{Gr}F$ is a Pareto minimum point for F, or a Pareto solution for (P), if there exists $\varepsilon\in(0,\infty]$ such that*

$$[F(B(\overline{x},\varepsilon))-\overline{y}]\cap -K=\{0\}.$$

If $\operatorname{int}K\neq\emptyset$, $(\overline{x},\overline{y})\in\operatorname{Gr}F$ is a weak Pareto minimum point for F, or a weak Pareto solution for (P), if there exists $\varepsilon\in(0,\infty]$ such that

$$[F(B(\overline{x},\varepsilon))-\overline{y}]\cap -\operatorname{int}K=\emptyset.$$

Obviously, in the above definitions, the case $\varepsilon\in(0,\infty)$ corresponds to the local minima, while the case $\varepsilon=+\infty$ describes the global solutions. We mention that the main results of this work apply to both situations.

It is easy to see that $(\overline{x},\overline{y})$ is a minimum for F (in any of the above senses) iff it is a minimum of the same type for the epigraphical set-valued map $\tilde{F}:X\rightrightarrows Y$, $\tilde{F}(x)=F(x)+K$.

Consider now a sequence (F_n) of set-valued mappings acting between X and Y. We associate the sequence of optimization problems, with respect to the same order $\leq_K$, as

$$(P_n)\quad \text{minimize } F_n(x), \text{ subject to } x\in X.$$

The main problem we discuss is the following one: having a sequence $(x_n,y_n)\in\operatorname{Gr}F_n$ of Pareto minima for (P_n) (for all n) such that $(x_n,y_n)\to(x,y)\in\operatorname{Gr}F$, what can we say about the point (x,y) in relation with problem (P) when (F_n) are, in a sense, approximations of F?

A well known fact is that, in general, (x,y) is not a Pareto minimum, even under nice convergence properties of (F_n) towards F. Basically, we propose ourselves to describe some general situations when the approximation properties of the sequence (F_n) ensure that (x,y) is a critical point for (P).

We present next the first of the main results, and for this consider the set-valued mappings (F_n), F as the objectives of the problems (P_n) and (P) introduced before.

Theorem 46 *Suppose that* X, Y *are Asplund spaces, and take* $(x, y) \in \operatorname{Gr} F$. *Consider a sequence* $(x_n, y_n) \to (x, y)$ *such that* $(x_n, y_n) \in \operatorname{Gr} F_n$ *is a minimum of radius* $\varepsilon_n > 0$ *for* F_n *for all* n. *Assume that:*

(i) $\operatorname{Gr} F$ *is locally closed at* (x, y);

(ii) K *is (SNC) at* 0, *or* F^{-1} *is (PSNC) at* (y, x);

(iii) $\liminf \varepsilon_n > 0$;

(iv) there exists a function $\varphi : [0, \infty) \to [0, \infty)$ *with* $\lim_{t \to 0} \varphi(t) = \varphi(0) = 0$ *such that, for all* n, *and for all small* α

$$\tilde{F}(B(x, \alpha)) \subset \tilde{F}_n(B(x, \alpha + \varphi(\alpha))). \tag{31}$$

Then there exists $v^* \in K^+ \setminus \{0\}$ *such that*

$$0 \in D^* F(x, y)(v^*),$$

that is, (x, y) *is a critical point of* F.

The second situation we study uses a new notion of openness, which reads as follows: given a multifunction $F : X \rightrightarrows Y$, a cone $K \subset Y$, a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, and two constants $\alpha, \beta > 0$, one says that F is $(\alpha, \beta)-$open with respect to K at $(\overline{x}, \overline{y})$ if there exists $\varepsilon > 0$ such that, for any $\rho \in (0, \varepsilon)$,

$$B(\overline{y}, \alpha\rho) \subset F(B(\overline{x}, \rho)) + K \cap B(0, \beta\rho). \tag{32}$$

First, we formulate a result concerning the stability of openness with respect to a cone at Lipschitz perturbations in the global case.

Theorem 47 *Let* K *be a closed convex cone, and* $F, G : X \rightrightarrows Y$ *be two multifunctions such that* $\operatorname{Gr} F$ *and* $\operatorname{Gr} G$ *are locally closed. Suppose that* $\operatorname{Dom}(F + G)$ *is nonempty and let* $\alpha > 0$, $\beta > 0$ *and* $\gamma > 0$ *be such that* $\alpha > \beta$. *If* F *is* $(\alpha, \gamma)-$*open with respect to* K *at every point of its graph, and if* G *is* $\beta-$*Aubin at every point of its graph, then* $F + G$ *is* $(\alpha - \beta, \gamma)-$*open with respect to* K *at every point of its graph.*

A similar result, formulated for the local case, holds, which involves the additional property of sum-stability of the pair (F, G) around the reference point.

The main result in this case is given by the next theorem.

Theorem 48 *Suppose that* X, Y *are Asplund spaces, and take* $(x, y) \in \operatorname{Gr} F$. *Take* $G_n : X \rightrightarrows Y$ *and consider a sequence* $(x_n, y_n) \to (x, y)$ *such that* $(x_n, y_n) \in \operatorname{Gr}(F + G_n)$ *is a minimum or* $F + G_n$ *for all* n. *Suppose that:*

(i) $\operatorname{Gr} F$ *is locally closed at* (x, y) *and for all* n, $\operatorname{Gr} G_n$ *is locally closed at every point from its graph close to* $(x, 0)$*;*

(ii) K *is (SNC) at* 0 *or* F^{-1} *is (PSNC) at* (y, x);

(iii) for all n, *there is* $\beta_n > 0$ *such that* G_n *in* β_n-*Aubin around every point from its graph close to* $(x, 0)$, *and* $\beta_n \to 0$;

(iv) for all n, *the pair* (F, G_n) *is locally sum-stable around* $(x, y, 0)$.

Then there exists $v^* \in K^+ \setminus \{0\}$ *such that*

$$0 \in D^* F(x, y)(v^*),$$

that is, (x, y) *is a critical point of* F.

The paper [3] aims at introducing and studying directional notions of Pareto efficiencies. Let $K \subset Y$ be a proper (that is, $K \neq \{0\}$, $K \neq Y$) convex cone (we do not suppose that K is pointed, in general).

Take $F : X \rightrightarrows Y$ as a set-valued mapping, and let us consider the following geometrically constrained optimization problem with multifunctions:

$$(P_{F,A}) \quad \text{minimize } F(x), \text{ subject to } x \in A,$$

where $A \subset X$ is a closed nonempty set.

Usually, the minimality is understood in the Pareto sense: point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a local Pareto minimum point for F on A if there exists a neighborhood U of $\overline{x}$ such that

$$(F(U \cap A) - \overline{y}) \cap -K \subset K; \tag{33}$$

if $\operatorname{int} K \neq \emptyset$, the point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a local weak Pareto minimum point for F on A if there exists a neighborhood U of $\overline{x}$ such that

$$(F(U \cap A) - \overline{y}) \cap -\operatorname{int} K = \emptyset.$$

Let $L \subset S_X$ be a nonempty closed set.

W introduce and to study the following concept.

Definition 49 *One says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a local directional Pareto minimum point for F on A with respect to (the set of directions) L if there exists a neighborhood U of $\overline{x}$ such that*

$$(F(U \cap A \cap (\overline{x} + \operatorname{cone} L)) - \overline{y}) \cap -K \subset K. \tag{34}$$

If one compares this relation to (33), then one observes that this concept corresponds to the situation where the restriction has the special form (depending on the reference point) $A \cap (\overline{x} + \operatorname{cone} L)$. Of course, when $A = X$ in (34) then one says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a local directional Pareto minimum point for F with respect to L. Now, the concept of local directional Pareto maximum is obtained in an obvious way.

If $\operatorname{int} K \neq \emptyset$, one defines as well the weak counterpart of the above notion.

Definition 50 *One says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a local weak directional Pareto minimum point for F on A with respect to (the set of directions) L if there exists a neighborhood U of $\overline{x}$ such that*

$$(F(U \cap A \cap (\overline{x} + \operatorname{cone} L)) - \overline{y}) \cap -\operatorname{int} K = \emptyset.$$

In all these notions, if one takes $U = X$, then we get the corresponding global concepts.

In order to get necessary optimality conditions for this problem we consider an appropriate directional tangent cone.

Definition 51 *Let $A \subset X$ be a nonempty set and $L \subset S_X$ be a nonempty closed set. Then the Bouligand tangent cone to A at $\overline{x} \in A$ with respect to L is the set*

$$T_B^L(A, \overline{x}) := \left\{ u \in X \mid \exists (u_n) \overset{\operatorname{cone} L}{\longrightarrow} u, \exists (t_n) \overset{(0,\infty)}{\longrightarrow} 0 \text{ such that for all } n,\ \overline{x} + t_n u_n \in A \right\},$$

where $(u_n) \overset{\operatorname{cone} L}{\longrightarrow} u$ means $(u_n) \longrightarrow u$ and $(u_n) \subset \operatorname{cone} L$, and similarly for $(t_n) \overset{(0,\infty)}{\longrightarrow} 0$.

Obviously, this is an adaptation of the concept of Bouligand tangent cone to A at $\overline{x}$ defined as

$$T_B(A, \overline{x}) := \left\{ u \in X \mid \exists (u_n) \to u, \exists (t_n) \overset{(0,\infty)}{\longrightarrow} 0 \text{ such that for all } n,\ \overline{x} + t_n u_n \in A \right\},$$

and, in fact,

$$T_B^L(A, \overline{x}) = T_B(A \cap (\overline{x} + \operatorname{cone} L), \overline{x}).$$

Definition 52 *Let $F : X \rightrightarrows Y$ be a set-valued map, $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ and $L \subset S_X$, $M \subset S_Y$ be nonempty closed sets. The Bouligand derivative of F at $(\overline{x}, \overline{y})$ with respect to L and M is the set-valued map $D_B^{L,M} F(\overline{x}, \overline{y}) : X \rightrightarrows Y$ defined by the relation $v \in D_B^{L,M} F(\overline{x}, \overline{y})(u)$ iff there are $(u_n) \overset{\operatorname{cone} L}{\longrightarrow} u, (v_n) \overset{\operatorname{cone} M}{\longrightarrow} v, (t_n) \overset{(0,\infty)}{\longrightarrow} 0$ such that for all n,*

$$\overline{y} + t_n v_n \in F(\overline{x} + t_n u_n).$$

Let $F : X \rightrightarrows Y$ be a set-valued mapping and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, $\emptyset \neq L \subset S_X$, $\emptyset \neq M \subset S_Y$.

What we need in the sequel is the following concept of directional calmness. One says that F is directionally calm at $(\overline{x}, \overline{y})$ with respect to L and M if there are $\alpha > 0$ and some neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that for every $x \in U$,

$$\sup_{y \in F(x) \cap V} T_M(y, F(\overline{x})) \leq \alpha T_L(\overline{x}, x). \tag{35}$$

We use the convention $\sup\limits_{x \in \emptyset} T_L(x, \Omega) := 0$ for every nonempty set $\Omega \subset X$.

As usual (see [6, Section 3H]), for a calmness concept for F, it is natural to have a metric subregularity notion such that the former property for F^{-1} to be equivalent to the latter property for F. In our setting, this corresponding concept reads as follows: one says that F is directionally metric subregular at $(\overline{x}, \overline{y})$ with respect to L and M if there exist $\alpha > 0$ and some neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that for every $x \in U$,

$$T_L(x, F^{-1}(\overline{y})) \leq \alpha T_M(\overline{y}, F(x) \cap V). \tag{36}$$

The expected equivalence is described in the following result.

Proposition 53 *The set-valued map F is directionally metric subregular at $(\overline{x}, \overline{y})$ with respect to L and M iff F^{-1} is directionally calm at $(\overline{y}, \overline{x})$ with respect to M and L.*

Consider now the situation when $G : X \rightrightarrows Z$ is a set-valued map, $Q \subset Z$ is a closed convex and pointed cone and the set of restrictions for $(P_{F,A})$ is $A := \{x \in X \mid 0 \in G(x) + Q\}$. Define the set-valued map $\mathcal{E}_G : X \rightrightarrows Z$, $\mathcal{E}_G(x) = G(x) + Q$. Then we have the next proposition.

Proposition 54 *Suppose that $\operatorname{int} K \neq \emptyset$ and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a local weak directional Pareto minimum point for F on $A := \mathcal{E}_G^{-1}(0)$ with respect to a closed nonempty set $L \subset S_X$. Consider $\overline{z} \in G(\overline{x}) \cap -Q$ and $\emptyset \neq N \subset S_Z$ a closed set. Moreover, suppose that $Q \cap S_Z \subset N$, $\operatorname{cone} L$ and $\operatorname{cone} N$ are convex, and $\mathcal{E}_G$ is directionally metric subregular at $(\overline{x}, 0)$ with respect to L and N. Then*

$$\left\{(v, w) \mid \exists u \in X, v \in D_D F(\overline{x}, \overline{y})(u), w \in D_B^{L,N} G(\overline{x}, \overline{z})(u)\right\} \cap (-\operatorname{int} K \times -Q) = \emptyset.$$

Let us to specialize, in two steps, the ideas above to the classical smooth case of optimization problems with single-valued maps. First, suppose that $F := f$ and $G := g$ are continuously Fréchet differentiable functions. Then taking a point $\overline{x} \in A = \mathcal{E}_G^{-1}(0)$ it is easy to see that for all $u \in X$, $D_D f(\overline{x})(u) = \{\nabla f(\overline{x})(u)\}$, while

$$D_B^{L,S_Z} g(\overline{x})(u) = \begin{cases} \{\nabla g(\overline{x})(u)\}, \text{ if } u \in \operatorname{cone} L \\ \emptyset, \text{ if } u \notin \operatorname{cone} L. \end{cases}$$

Then we get the following Fritz John and Karush-Kuhn-Tucker type result.

Theorem 55 *Suppose that $\operatorname{int} K \neq \emptyset$ and $\overline{x} \in A := \mathcal{E}_g^{-1}(0)$ is a local weak directional Pareto minimum point for f on A with respect to L. Moreover, suppose that $\operatorname{cone} L$ is convex, and $\mathcal{E}_g$ is directionally metric subregular at $(\overline{x}, 0)$ with respect to L and S_Z. Then, in either of the following conditions:*

(i) $\operatorname{int} Q \neq \emptyset$ or $\operatorname{int}\{(\nabla f(\overline{x})(u), \nabla g(\overline{x})(u)) \mid u \in \operatorname{cone} L\} \neq \emptyset$;

(ii) Y and Z are finite dimensional spaces,

there exist $y^* \in K^+$, $z^* \in Q^+$, $(y^*, z^*) \neq 0$ such that for every $u \in \operatorname{cone} L$,

$$(y^* \circ \nabla f(\overline{x}) + z^* \circ \nabla g(\overline{x}))\,(u) \geq 0.$$

If, moreover, there exists $u \in \operatorname{cone} L$ such that $\nabla g(\overline{x})(u) \in \operatorname{int} Q \neq \emptyset$ or $\nabla g(\overline{x})(\operatorname{cone} L) = Z$ then $y^* \neq 0$.

Furthermore, we consider the case where $Y = \mathbb{R}^k$ $(k \geq 1)$, $Z = \mathbb{R}^p$ $(p \geq 1)$, $Q = \mathbb{R}_+^m \times \{0\}^n$ with $m + n = p$, and f, g are Fréchet differentiable. This means that we are dealing with a vectorial optimization problem with finitely many inequalities and equalities constraints. Let us denote by μ_i with $i \in \overline{1, m}$ the first m coordinates functions of g and by ν_j with $j \in \overline{1, n}$ the next n coordinates functions of g.

In this notation we have the next result.

Theorem 56 *Suppose that X is a Banach space, $\operatorname{int} K \neq \emptyset$ and $\overline{x} \in g^{-1}(-Q)$ is a local weak directional Pareto minimum point for f on $g^{-1}(-Q)$ with respect to L. Suppose that:*

(i) $\operatorname{cone} L$ is convex;

(ii) $\psi : X \times Z \to Z$, $\psi(x, z) := g(x) - z$ is metrically subregular at $(\overline{x}, g(\overline{x}), 0)$ with respect to $(\overline{x} + \operatorname{cone} L) \times -Q$;

(iii) $\nabla \nu\,(\overline{x})\,(X) = \mathbb{R}^n$, where $\nu := (\nu_1, \nu_2, ..., \nu_n)$;

(iv) there exists $\overline{u} \in \operatorname{int} \operatorname{cone} L$ such that $\nabla \mu_i\,(\overline{x})\,(\overline{u}) < 0$ for any $i \in I(\overline{x}) := \{i \in \overline{1, m} \mid \mu_i(\overline{x}) = 0\}$ and $\nabla \nu\,(\overline{x})\,(\overline{u}) = 0$.

Then there exist $y^* \in K^+\backslash\{0\}$, $\lambda_i \geq 0$ *for* $i \in \overline{1,m}$ *and* $\tau_j \in \mathbb{R}$ *for* $j \in \overline{1,n}$ *such that*

$$0 \in y^* \circ \nabla f(\overline{x}) + \sum_{i=1}^{m} \lambda_i \nabla \mu_i(\overline{x}) + \sum_{j=1}^{n} \tau_j \nabla \nu_j(\overline{x}) + L^- \tag{37}$$

and

$$\lambda_i \mu_i(\overline{x}) = 0, \forall i \in \overline{1,m}. \tag{38}$$

Using limiting generealized differentiation objects, one have the next result for the general case of problem $(P_{F,A})$.

Theorem 57 *Let* $A \subset X$ *and* $L \subset S_X$ *be nonempty closed sets and* $F : X \rightrightarrows Y$ *be a set-valued map with* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ *such that* $\operatorname{Gr} F$ *is closed around* $(\overline{x}, \overline{y})$*. Suppose that the following assertions hold:*

(i) F *is Lipschitz-like around* $(\overline{x}, \overline{y})$*;*

(ii) $K \backslash -K \neq \emptyset$ *and* K *is (SNC) at* 0*;*

(iii) the sets A *and* $\overline{x} + \operatorname{cone} L$ *are allied at* $\overline{x}$*.*

If $(\overline{x}, \overline{y})$ *is a local directional Pareto minimum point for* F *on* A *with respect to the set of directions* L*, then there exists* $y^* \in K^+ \backslash \{0\}$ *such that*

$$0 \in D^* F(\overline{x}, \overline{y})(y^*) + N(A, \overline{x}) + N(\operatorname{cone} L, 0).$$

In the paper [18] we introduce a concept of Pareto efficiency with respect to two sets of directions and we investigate it in detail from the point of view of necessary optimality conditions, employing the methods of variational analysis and generalized differentiation.

Consider X and Y Banach spaces over the real field $\mathbb{R}$, $K \subset Y$ a proper (that is, $K \neq \{0_Y\}$, $K \neq Y$) convex cone, $F : X \rightrightarrows Y$ a set-valued map and $A \subset X$ a nonempty set. We denote by $\operatorname{int} A$ the topological interior of A and by $\operatorname{cone} A$ its conic hull, while X^* stands for the topological dual of X.

In the following, we introduce the principal concept of directional Pareto minimality studied in paper [18]. For this we consider $L \subset S_X$ and $M \subset S_Y$ to be nonempty and closed sets, where S_X stands for the unit sphere in the space X.

Definition 58 *(i) One says that* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ *is a local directional Pareto minimum point for* F *on* A *with respect to* $L \subset S_X$ *and* $M \subset S_Y$ *if there exists a neighborhood* U *of* $\overline{x}$ *such that*

$$[F(U \cap A \cap (\overline{x} + \operatorname{cone} L)) \cap (\overline{y} - \operatorname{cone} M) - \overline{y}] \cap -K \subset K,$$

where we denote by $\operatorname{Gr} F := \{(x, y) \in X \times Y \mid y \in F(x)\}$ *the graph of* F*.*

(ii) If $\operatorname{int} K \neq \emptyset$*, one says that* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ *is a local weak directional Pareto minimum point for* F *on* A *with respect to* $L \subset S_X$ *and* $M \subset S_Y$ *if there exists a neighborhood* U *of* $\overline{x}$ *such that*

$$[F(U \cap A \cap (\overline{x} + \operatorname{cone} L)) \cap (\overline{y} - \operatorname{cone} M) - \overline{y}] \cap - \operatorname{int} K = \emptyset.$$

If $A = X$, then we have unconstrained efficiencies and we omit to write "on A". Also, if $U = X$, then we get the corresponding global concepts. Mention that, if $M = S_Y$, the above directional Pareto minimality notion coincide with the already existing concept of Pareto efficiency with respect to one set of directions studied in [3]. Moreover, if in addition $L = S_X$, then both directional concepts reduce to the classical concept of Pareto efficiency.

The main purpose of the paper is to get necessary optimality conditions for the directional minimality concept introduced in Definition 58 using generalized differentiation objects lying in both primal and dual spaces. In order to do this, we follow the next two steps. First, we prove an incompatibility result between a directional openness concept and the Pareto minimality introduced in [18]. A second step, motivated by the first one, is to obtain sufficient conditions for the directional openness working, on one hand, with Bouligand directional derivative, and on the other hand, with Mordukhovich coderivative. To achieve the sufficient conditions of directional openness on dual spaces, we prove as well a version of the Ekeland Variational Principle adapted to the situation we have to study.

Further, we give here only the result concerning the necessary optimality conditions obtained on dual spaces, in terms of the Mordukhovich coderivative of the objective map.

Theorem 59 *Let X, Y be finite dimensional spaces, $F : X \rightrightarrows Y$ be a multifunction with closed graph and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ be a local directional Pareto minimum point for F with respect to L and $\{u\} \subset S_Y$. Suppose L is closed,* $\operatorname{cone} L$ *is convex, $u \in \operatorname{int} K \cap S_Y$ and F is Lipschitz-like around $(\overline{x}, \overline{y})$.*

Then, there are $x^* \in X^*$, $y^* \in K^+$ with $\langle x^*, l \rangle \geq 0$ for all $l \in L$ and $\langle y^*, u \rangle = 1$ such that

$$x^* \in D^* F(\overline{x}, \overline{y})(y^*).$$

The paper [2] continues the investigation from [3] and [18] by describing penalization techniques and optimality conditions for directional minimality within the framework of functionally constrained problems. In order to have more flexibility for the used concepts we propose the following definition.

Definition 60 *Let $L \subset S_X$ and $M \subset S_Y$ be nonempty closed sets. One says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a $(L, M)-$local directional Pareto minimum point for F on A (or for problem $(P_{F,A})$) if there exists a neighborhood U of $\overline{x}$ such that*

$$(F(U \cap A \cap [\overline{x} + \operatorname{cone} L]) - \overline{y}) \cap - \operatorname{cone} M \subset \operatorname{cone} M.$$

We denote the set of $(L, M)-$local directional Pareto minimum point for F on A by $\operatorname{Min}(F, A; L, M)$.

The first penalization result we record reads as follows.

Theorem 61 *Let $L \subset S_X$ and $M \subset S_Y$ be nonempty closed sets such that $M \cap -M = \emptyset$ and* $\operatorname{cone} M$ *is convex. Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ be a $(L, M)-$local directional Pareto minimum point for problem $(P_{F,A})$. Suppose that:*

(i) $A \cap [\overline{x} + \operatorname{cone} L]$ is locally closed at $\overline{x}$;

(ii) there exist $\ell > 0$, $e \in M$ and U a neighborhood of $\overline{x}$ such that, for every $(x', x'') \in (U \cap [\overline{x} + \operatorname{cone} L]) \times (U \cap A \cap [\overline{x} + \operatorname{cone} L])$,

$$F(x') + \ell \|x' - x''\| e \subset F(x'') + \operatorname{cone} M.$$

Then, for every $\ell' > \ell$, $(\overline{x}, \overline{y})$ is $(L, M) -$local directional Pareto minimum point for the unconstrained problem

$$\min \ F(x) + \ell' d(x, A \cap [\overline{x} + \operatorname{cone} L]) e.$$

The next penalization result we propose refers to the case of generalized functional constraints. More precisely, we consider the problem (P) where one replaces the set A with the set

$$\{x \in X \mid 0 \in G(x) + Q\} = G^{-1}(-Q),$$

where $G : X \rightrightarrows Z$ is a set-valued maps and $Q \subset Z$ is a proper convex closed cone.

Theorem 62 *Let $L \subset S_X$ be a nonempty closed set such that* $\operatorname{cone} L$ *is convex, and $M \subset S_Y$ be a nonempty closed set such that $M \cap -M = \emptyset$ and* $\operatorname{cone} M$ *is convex. Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (G^{-1}(-Q) \times Y)$ be a $(L, M)-$local directional Pareto minimum point for problem (P_f). Suppose that:*

(i) there exist $\ell > 0, e \in M$ and U a neighborhood of $\overline{x}$ such that, for every $(x', x'') \in (U \cap [\overline{x} + \operatorname{cone} L]) \times \left(U \cap G^{-1}(-Q) \cap [\overline{x} + \operatorname{cone} L]\right)$,

$$F(x') + \ell \|x' - x''\| e \subset F(x'') + \operatorname{cone} M.$$

(ii) $\tilde{G}^{-1}$ is is directionally calm at $(0, \overline{x})$ with respect to S_Q and L with constant $\alpha > 0$.

Then, for any $\alpha' > \alpha$ the point $((\overline{x}, 0), \overline{y})$ is a $((L, S_Q), M)-$local directional Pareto minimum point on $\operatorname{Gr} \tilde{G}$ for the mapping

$$(x, z) \rightrightarrows F(x) + \ell\alpha' \|z\| e.$$

By means of generalized differentiation the optimality conditions we get are as follows.

Theorem 63 *Let X, Y, Z be finite dimensional, $L \subset S_X$, $M \subset S_Y$ be nonempty closed sets such that* $\operatorname{cone} L, \operatorname{cone} M$ *are convex and $M \cap -M = \emptyset$. Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (G^{-1}(-Q) \times Y)$ and take $\bar{q} \in Q \cap -G(\overline{x})$. Assume that* $\operatorname{int} \operatorname{cone} M \neq \emptyset$ *and* $\operatorname{int} Q \neq \emptyset$ *and consider $(v, q) \in \operatorname{int} \operatorname{cone} M \times \operatorname{int} Q$. Suppose that*

(i) $\operatorname{Gr} F$ *and* $\operatorname{Gr} G$ *are locally closed at $(\overline{x}, \overline{y})$ and $(\overline{x}, -\bar{q})$, respectively;*

(ii) F and G are Lipschitz-like around $(\overline{x}, \overline{y})$ and $(\overline{x}, -\bar{q})$, respectively.

If $(\overline{x}, \overline{y})$ is a $(L, M)-$local directional Pareto minimum point for problem $(P_{F,A})$, then there exist $x^* \in L^+$, $y^* \in (\operatorname{cone} M)^+$, $z^* \in Q^+$ such that $(y^*, z^*)(v, q) = 1$ and

$$x^* \in D^*F(\overline{x}, \overline{y})(y^*) + D^*G(\overline{x}, -\bar{q})(z^*).$$

In the paper [45] we considered general quadratic optimization problems with equality and/or inequality quadratic constraints. Our motivation was to compare the obtained results with those established by Gao and some of his collaborators using Gao's Canonical duality theory, denoted CDT; see [20] for a survey on CDT. Our study is done using the usual Lagrange function and an associated dual function suggested by CDT. Besides the fact that our results are more general, at least because we do not ask the strict positivity of all Lagrange multipliers corresponding to the nonlinear constraints, we provide counterexamples to several results in the literature.

So, we consider the problem

(P_J) $\quad \min q_0(x)$ s.t. $x \in X_J$,

where

$$X_J := \{x \in \mathbb{R}^n \mid [\forall j \in J : q_j(x) = 0] \ \wedge \ [\forall j \in J^c : q_j(x) \leq 0]\},$$

with $J \subset \overline{1,m}$, $J^c := \overline{1,m} \setminus J$ and $q_k(x) := \frac{1}{2}\langle x, A_k x\rangle - \langle b_k, x\rangle + c_k$ for $x \in \mathbb{R}^n$ in which $A_k \in \mathbb{R}^{n \times n}$ is symmetric, $b_k \in \mathbb{R}^n$ (seen as column vector) and $c_k \in \mathbb{R}$ for $k \in \overline{0,m}$ (with $c_0 := 0$). For later use we introduce also the set

$$\Gamma_J := \{(\lambda_1, ..., \lambda_m) \in \mathbb{R}^m \mid \lambda_j \geq 0 \ \forall j \in J^c\}.$$

To the family $(q_k)_{k \in \overline{0,m}}$ we associate the Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$L(x, \lambda) := q_0(x) + \sum\nolimits_{j=1}^m \lambda_j q_j(x) = \tfrac{1}{2}\langle x, A(\lambda)x\rangle - \langle x, b(\lambda)\rangle + c(\lambda), \tag{39}$$

where

$$A(\lambda) := \sum\nolimits_{k=0}^m \lambda_k A_k, \quad b(\lambda) := \sum\nolimits_{k=0}^m \lambda_k b_k, \quad c(\lambda) := \sum\nolimits_{k=0}^m \lambda_k c_k, \tag{40}$$

with $\lambda_0 := 1$ and $\lambda := (\lambda_1, ..., \lambda_m)^T \in \mathbb{R}^m$. We consider also the sets

$$Y_0 := \{\lambda \in \mathbb{R}^m \mid \det A(\lambda) \neq 0\}, \quad Y_{\mathrm{col}} := \{\lambda \in \mathbb{R}^m \mid b(\lambda) \in \operatorname{Im} A(\lambda)\}, \tag{41}$$

$$Y^+(Y^-) := \{\lambda \in \mathbb{R}^m \mid A(\lambda) \succ (\prec) 0\}, \quad Y^+_{\mathrm{col}}(Y^-_{\mathrm{col}}) := \{\lambda \in Y_{\mathrm{col}} \mid A(\lambda) \succeq (\preceq) 0\}, \tag{42}$$

$$Y^J := \Gamma_J \cap Y_0, \quad Y^{J+} := \Gamma_J \cap Y^+, \quad Y^J_{\mathrm{col}} := \Gamma_J \cap Y_{\mathrm{col}}, \quad Y^{J+}_{\mathrm{col}} := \Gamma_J \cap Y^+_{\mathrm{col}}. \tag{43}$$

Clearly, $Y^+ \cup Y^- \subset Y_0 \subset Y_{\mathrm{col}}$, $Y^+ \subset Y^+_{\mathrm{col}}$ and $Y^- \subset Y^-_{\mathrm{col}}$. Of course, Y_0 is a (possibly empty) open set, while Y^+ and Y^- are open convex sets; Y_{col} is neither open, nor closed (in general). Unlike for Y^+, the convexity of Y^+_{col} is less obvious, but is true.

Of course, for every $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m$ we have that

$$\nabla_x L(x, \lambda) = A(\lambda) \cdot x - b(\lambda), \quad \nabla^2_{xx} L(x, \lambda) = A(\lambda), \quad \nabla_\lambda L(x, \lambda) = (q_j(x))_{j \in \overline{1,m}}.$$

Hence $L(\cdot, \lambda)$ is (strictly) convex for $\lambda \in Y^+_{\mathrm{col}}$ ($\lambda \in Y^+$) and (strictly) concave for $\lambda \in Y^-_{\mathrm{col}}$ ($\lambda \in Y^-$). Moreover, for $\lambda \in Y_0$ we have $\nabla_x L(x, \lambda) = 0$ iff $x = x(\lambda) := A(\lambda)^{-1} b(\lambda) := [A(\lambda)]^{-1} \cdot b(\lambda)$.

Let us consider now the (dual objective) function

$$D_L : Y_{\mathrm{col}} \to \mathbb{R}, \quad D_L(\lambda) := L(x, \lambda) \text{ with } A(\lambda)x = b(\lambda); \tag{44}$$

D_L is well defined because for $x_1, x_2 \in \mathbb{R}^n$ with $A(\lambda)x_1 = A(\lambda)x_2 = b(\lambda)$, we have that $L(x_2, \lambda) = L(x_1, \lambda)$. Clearly, $D_L(\lambda) = L(x(\lambda), \lambda)$ for $\lambda \in Y_0$. Of course

$$D_L(\lambda) = L\big(A(\lambda)^{-1} b(\lambda), \lambda\big) = -\tfrac{1}{2}\big\langle b(\lambda), A(\lambda)^{-1} b(\lambda)\big\rangle + c(\lambda) \quad \forall \lambda \in Y_0.$$

Lemma 64 *Let* $(\overline{x},\overline{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$ *be such that* $\nabla_x L(\overline{x},\overline{\lambda}) = 0$ *and* $\langle \overline{\lambda}, \nabla_\lambda L(\overline{x},\overline{\lambda}) \rangle = 0$*. Then* $\overline{\lambda} \in Y_{\mathrm{col}}$ *and*

$$q_0(\overline{x}) = L(\overline{x},\overline{\lambda}) = D_L(\overline{\lambda}). \tag{45}$$

In particular, if $(\overline{x},\overline{\lambda})$ *is a critical point of* L *[that is* $\nabla L(\overline{x},\overline{\lambda}) = 0$*], then* $\overline{x} \in X_{\overline{1,m}}$ *and (45) hold.*

Formula (45) is related to the so-called "complimentary-dual principle" (see [20, p. 13]) and sometimes is called the "perfect duality formula".

Observe that D_L is a C^∞ function on the open set Y_0. It follows that

$$\frac{\partial D_L(\lambda)}{\partial \lambda_j} = \tfrac{1}{2}\langle x(\lambda), A_j x(\lambda)\rangle - \langle b_j, x(\lambda)\rangle + c_j = q_j(x(\lambda)) \quad \forall j \in \overline{1,m},\ \lambda \in Y_0, \tag{46}$$

and so

$$\forall \lambda' \in Y_0 : [\nabla D_L(\lambda') = 0 \iff \nabla_\lambda L(x(\lambda'),\lambda') = 0 \iff \nabla L(x(\lambda'),\lambda') = 0].$$

Below we state the main result of [45], but first we introduce some notions suggested by the well known necessary optimality conditions for minimization problems with equality and inequality constraints. We say that $(\overline{x},\overline{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$ is a J-LKKT point of L if $\nabla_x L(\overline{x},\overline{\lambda}) = 0$ and

$$\overline{x} \in X_J \quad \wedge \quad \overline{\lambda} \in \Gamma_J \quad \wedge \quad [\forall j \in J^c : \overline{\lambda}_j q_j(\overline{x}) = 0]; \tag{47}$$

we say that $\overline{x} \in \mathbb{R}^n$ is a J-LKKT point for (P_J) if there exists $\overline{\lambda} \in \mathbb{R}^m$ such that $(\overline{x},\overline{\lambda})$ verifies (47); moreover, for D_L defined in (44), we say that $\overline{\lambda} \in Y_0$ is a J-LKKT point for D_L if

$$[\forall j \in J^c : \overline{\lambda}_j \geq 0 \quad \wedge \quad \tfrac{\partial D_L}{\partial \lambda_j}(\overline{\lambda}) \leq 0 \quad \wedge \quad \overline{\lambda}_j \cdot \tfrac{\partial D_L}{\partial \lambda_j}(\overline{\lambda}) = 0] \ \wedge \ [\forall j \in J : \tfrac{\partial D_L}{\partial \lambda_j}(\overline{\lambda}) = 0].$$

Of course, when $J = \overline{1,m}$, $(\overline{x},\overline{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$ is a J-LKKT point of L iff $\nabla L(\overline{x},\overline{\lambda}) = 0$, while $\overline{\lambda} \in Y_0$ is a J-LKKT point for D_L iff $\nabla D_L(\overline{\lambda}) = 0$. For $J = \emptyset$, the preceding conditions become the usual KKT conditions.

Proposition 65 *Let* $(\overline{x},\overline{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$*.*

(i) *Assume that* $(\overline{x},\overline{\lambda})$ *is a J-LKKT point of* L*. Then* $\overline{x}$ *is a J-LKKT point of* (P_J)*,* $\overline{x} \in X_J$*,* $\overline{\lambda} \in Y^J_{\mathrm{col}}$*, and (45) holds; moreover, if* $\overline{\lambda} \in Y^{J+}_{\mathrm{col}}$ *then*

$$q_0(\overline{x}) = \inf_{x \in X_J} q_0(x) = L(\overline{x},\overline{\lambda}) = \sup_{\lambda \in Y^{J+}_{\mathrm{col}}} D_L(\lambda) = D_L(\overline{\lambda}). \tag{48}$$

(ii) *Assume that* $(\overline{x},\overline{\lambda})$ *is a J-LKKT point of* L *with* $\overline{\lambda} \in Y_0$ *(or, equivalently,* $\overline{\lambda} \in Y^J$*). Then* $\overline{x} = A(\overline{\lambda})^{-1} b(\overline{\lambda})$*, and* $\overline{\lambda}$ *is a J-LKKT point of* D_L*; moreover,* $\overline{x}$ *is the unique global minimizer of* q_0 *on* X_J *provided* $\overline{\lambda} \in Y^{J+}$*. Conversely, assume that* $\overline{\lambda} \in Y_0$ *is a J-LKKT point of* D_L*. Then* $(\overline{x},\overline{\lambda})$ *is a J-LKKT point of* L*, where* $\overline{x} := A(\overline{\lambda})^{-1} b(\overline{\lambda})$*. Consequently,* (i) *and* (ii) *apply.*

(iii) *Assume that* $\overline{\lambda} \in Y^{J+}$*. Then*

$$D_L(\overline{\lambda}) = \sup_{\lambda \in Y^{J+}_{\mathrm{col}}} D_L(\lambda) \iff D_L(\overline{\lambda}) = \sup_{\lambda \in Y^{J+}} D_L(\lambda) \iff \overline{\lambda} \text{ is a J-LKKT point of } D_L.$$

Important cases are $J := \overline{1,m}$ (of equality constraints) and $J := \emptyset$ (of inequality constraints).

Remark. Jeyakumar, Rubinov and Wu (see [28, Prop. 3.2]) proved that $\overline{x}$ is a (global) solution of $(P_\emptyset)$ whenever there exists $\overline{\lambda} \in \mathbb{R}^m_+ \cap Y_{\mathrm{col}}$ which is a KKT point of L; this result was established previously by Hiriart-Urruty in [27, Th. 4.6] when $m = 2$.

In the paper [46] we study the following unconstrained minimization problem

(P) min $f(x)$ s.t. $x \in \mathbb{R}^n$

where $f := q_0 + V \circ q$ with $q(x) := (q_1(x),...,q_m(x))^T$, $q_i(x) := \frac{1}{2}\langle x, A_i x\rangle - \langle b_i, x\rangle + c_i$ for $x \in \mathbb{R}^n$ ($A_i \in \mathfrak{S}_n$, $b_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$), and $V \in \Gamma(\mathbb{R}^m)$, the class of proper convex lsc functions $g : \mathbb{R}^m \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. For notations and results on convex analysis in finite dimensions we refer to Rockafellar's book [39].

Consider the so called "total complementary function" $\Xi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ (see [21, p. 134]), associated to (P)

$$\Xi(x,\lambda) := L(x,\lambda) - V^*(\lambda) = \tfrac{1}{2}\langle x, A(\lambda)x\rangle - \langle b(\lambda), x\rangle + c(\lambda) - V^*(\lambda), \tag{49}$$

where L is the (usual) Lagrangian associated to $(q_k)_{k\in\overline{0,m}}$ (see (39)) and $A(\lambda)$, $b(\lambda)$, $c(\lambda)$ are defined in (40). Since $V^{**} := (V^*)^* = V$, from the definition of the conjugate of V^* and (49) we get

$$f(x) = \sup\nolimits_{\lambda\in\operatorname{dom} V^*} \Xi(x,\lambda) = \sup\nolimits_{\lambda\in\operatorname{ri}(\operatorname{dom} V^*)} \Xi(x,\lambda) \quad \forall x\in\mathbb{R}^n, \tag{50}$$

because for a proper convex function $g:\mathbb{R}^m\to\overline{\mathbb{R}}$ one has $g^* = (g+\iota_{\operatorname{ri}(\operatorname{dom} g)})^*$ (see [39, p. 259]). In the sequel the sets Y_0, Y^+, Y^-, $Y_{\rm col}$, $Y_{\rm col}^+$ and $Y_{\rm col}^-$ are those defined in (41)–(43).

We consider the (dual objective) function D associated to $(q_k)_{k\in\overline{0,m}}$ and V defined by

$$D : S_{\rm col} := Y_0 \cap \operatorname{dom} V^* \to \mathbb{R}, \quad D(\lambda) := \Xi(x,\lambda) \text{ with } A(\lambda)x = b(\lambda);$$

hence $D(\lambda) = D_L(\lambda) - V^*(\lambda)$ for $\lambda\in S_{\rm col}$.

Proposition 66 *Let $V\in\Gamma(\mathbb{R}^m)$ and $(\overline{x},\overline{\lambda})\in\mathbb{R}^n\times\operatorname{dom} V^*$.*

(i) *Assume that $\nabla_x\Xi(\overline{x},\overline{\lambda}) = 0$ and $q(\overline{x})\in\partial V^*(\overline{\lambda})$. Then $\overline{x}\in\operatorname{dom} f$, $\overline{\lambda}\in S_{\rm col} = Y_{\rm col}\cap\operatorname{dom} V^*$, $\overline{\lambda}\in\partial V(q(\overline{x}))$, and $f(\overline{x}) = \Xi(\overline{x},\overline{\lambda}) = D(\overline{\lambda})$.*

(ii) *Moreover, assume that $A(\overline{\lambda})\succeq 0$. Then $\overline{\lambda}\in S_{\rm col}^+ := Y_{\rm col}^+\cap\operatorname{dom} V^*$ and*

$$f(\overline{x}) = \inf\nolimits_{x\in\operatorname{dom} f} f(x) = \Xi(\overline{x},\overline{\lambda}) = \sup\nolimits_{\lambda\in S_{\rm col}^+} D(\lambda) = D(\overline{\lambda}); \tag{51}$$

furthermore, if $\overline{\lambda}\in S^+ := Y^+\cap\operatorname{dom} V^*$, then $\overline{x}$ is the unique global solution of problem (P).

In many papers by DY Gao and his collaborators one speaks about "triality theorems" in which, besides the minimax result established for the case $A(\overline{\lambda})\succeq 0$ (see Proposition 66), one obtains also "bi-duality" results established for $A(\overline{\lambda})\prec 0$, that is $\overline{x}$ and $\overline{\lambda}$ are simultaneously local minimizers (maximizers) for f on $\operatorname{dom} f$ and for D on $S^- := Y^-\cap\operatorname{dom} V^*$, respectively. With a (counter-) example we showed in [46] that such triality results are not valid for general $V\in\Gamma(\mathbb{R}^m)$, even for $n = m = 1$.

In DY Gao's works published after 2011 the "triality theorems" are established for V a twice differentiable strictly convex function. Our aim in the sequel is to study the problems of "bi-duality" for a special class of functions V. First we establish an auxiliary result on positive semidefinite operators in Euclidean spaces needed for getting our "bi-duality" results.

Proposition 67 *Let X, Y be nontrivial Euclidean spaces and $H : Y\to X$ be a linear operator with $H^* : X\to Y$ its adjoint. Consider $Q := HH^* := H\circ H^*$, $R := H^*H$, and*

$$\varphi : X\to\mathbb{R}, \quad \varphi(x) := \|H^*x\|^2, \qquad \psi : Y\to\mathbb{R}, \quad \psi(y) := \|Hy\|^2.$$

Then the following assertions hold:

(a) *Q and R are self-adjoint positive semi-definite operators, $\ker Q = \ker H^*$, $\operatorname{Im} Q = \operatorname{Im} H$, $\ker R = \ker H$, $\operatorname{Im} R = \operatorname{Im} H^*$; consequently, $H = 0 \Leftrightarrow Q = 0 \Leftrightarrow R = 0$.*

(b) *Setting $S_X := \{x\in X \mid \|x\| = 1\}$, one has $\alpha = \beta$, where*

$$\alpha := \max_{x\in S_X} \varphi(x) = \max\{\lambda\in\mathbb{R} \mid \exists x\in X\setminus\{0\} : Qx = \lambda x\},$$
$$\beta := \max_{y\in S_Y} \psi(y) = \max\{\lambda\in\mathbb{R} \mid \exists y\in Y\setminus\{0\} : Ry = \lambda y\}.$$

(c) *If $H\neq 0$, then $\operatorname{Im} Q\neq\{0\}$, $\operatorname{Im} R\neq\{0\}$, and $\gamma = \delta > 0$, where*

$$\gamma := \min_{x\in S_X\cap\operatorname{Im} Q} \varphi(x) = \min\{\lambda > 0 \mid \exists x\in X\setminus\{0\} : Qx = \lambda x\},$$
$$\delta := \min_{x\in S_Y\cap\operatorname{Im} R} \psi(y) = \min\{\lambda > 0 \mid \exists y\in Y\setminus\{0\} : Ry = \lambda y\}.$$

(d) *The following implications hold:*

$$\min_{x\in S_X} \varphi(x) = 0 \Longleftrightarrow \ker Q\neq\{0\} \Longleftrightarrow \operatorname{Im} Q\neq X \Longleftrightarrow \operatorname{Im} H\neq X,$$
$$\min_{y\in S_Y} \psi(y) = 0 \Longleftrightarrow \ker R\neq\{0\} \Longleftrightarrow \operatorname{Im} R\neq Y \Longleftrightarrow \ker H\neq\{0\}.$$

Let us denote by $\Gamma_{sc} := \Gamma_{sc}(\mathbb{R}^m)$ the class of those $g \in \Gamma(\mathbb{R}^m)$ which are essentially strictly convex and essentially smooth, that is the class of proper lsc convex functions of Legendre type (see [39, Section 26]), and $\Gamma_{sc}^2 := \Gamma_{sc}^2(\mathbb{R}^m)$ the class of those $g \in \Gamma_{sc}$ which are twice differentiable on $\operatorname{int}(\operatorname{dom} g)$ with $\nabla^2 g(y) \succ 0$ for $y \in \operatorname{int}(\operatorname{dom} g)$. Note that for $g \in \Gamma_{sc}$ we have: $g^* \in \Gamma_{sc}$, $\operatorname{dom} \partial g = \operatorname{int}(\operatorname{dom} g)$, and g is differentiable on $\operatorname{int}(\operatorname{dom} g)$; moreover, $\nabla g : \operatorname{int}(\operatorname{dom} g) \to \operatorname{int}(\operatorname{dom} g^*)$ is bijective and continuous with $(\nabla g)^{-1} = \nabla g^*$. Observe that for $g \in \Gamma_{sc}^2$ one has $g^* \in \Gamma_{sc}^2$ and

$$\nabla^2 g^*(\lambda) = \left(\nabla^2 g\left((\nabla g)^{-1}(\lambda)\right)\right)^{-1} \quad \forall \lambda \in \operatorname{int}(\operatorname{dom} g^*).$$

In the sequel $X_0 := \{x \in \mathbb{R}^n \mid q(x) \in \operatorname{int}(\operatorname{dom} V)\}$, and we assume that $V \in \Gamma_{sc}$ (and so $V^* \in \Gamma_{sc}$). It follows that

$$\left\langle u, \nabla^2 f(x) u\right\rangle = \left\langle u, \left[A_0 + \sum\nolimits_{i=1}^m \frac{\partial V}{\partial y_i}(q(x)) \cdot A_i\right] u\right\rangle + \left\langle v_u, \nabla^2 V(q(x)) v_u\right\rangle \tag{52}$$

for all $x \in X_0$ and $u \in \mathbb{R}^n$, where $v_u := (\langle u, A_1 x - b_1\rangle, ..., \langle u, A_m x - b_m\rangle)^T$, and

$$\left\langle v, \nabla^2 D(\lambda) v\right\rangle = -\left\langle A_v x(\lambda) - b_v, A(\lambda)^{-1}\left(A_v x(\lambda) - b_v\right)\right\rangle - \left\langle v, \nabla^2 V^*(\lambda) v\right\rangle$$

for all $v \in \mathbb{R}^m$ and $\lambda \in S_0$, where $x(\lambda) := A(\lambda)^{-1} b(\lambda)$, $A_v := \sum_{i=1}^m v_j A_j$, $b_v := \sum_{j=1}^m v_j b_j$.

Assume that $(\overline{x}, \overline{\lambda}) \in X_0 \times S^-$ is a critical point of Ξ, where $S^- := Y^- \cap \operatorname{dom} V^*$; then $\overline{\lambda} = \nabla V(q(\overline{x}))$. Because $A(\overline{\lambda}) \prec 0$ and $\nabla^2 V(q(\overline{x})) \succ 0$ there exist non-singular matrices $E \in \mathfrak{M}_n$ and $F \in \mathfrak{M}_m$ such that $-A(\overline{\lambda}) = E^* E$ and $\nabla^2 V(q(\overline{x})) = F^* F$. Setting $d_i := (E^{-1})^*(A_i \overline{x} - b_i) \in \mathbb{R}^n$ $(i \in \overline{1, m})$, $u' := Eu$ for $u \in \mathbb{R}^n$ and $v' := (F^{-1})^* v$ for $v \in \mathbb{R}^m$, we obtain that

$$\left\langle u, \nabla^2 f(\overline{x}) u\right\rangle = \left\|H^* u'\right\|^2 - \left\|u'\right\|^2, \quad \left\langle v, \nabla^2 D(\overline{\lambda}) v\right\rangle = \left\|H v'\right\|^2 - \left\|v'\right\|^2,$$

where $H := J \circ F^*$ with $Jv := \sum_{i=1}^m v_i d_i$ for $v \in \mathbb{R}^m$. Because E and F are non-singular, for $\rho \in \{>, \geq, <, \leq\}$ and $\rho' \in \{\succ, \succeq, \prec, \preceq\}$ with the natural correspondence, for $S_p := \{z \in \mathbb{R}^p \mid \|y\| = 1\} = S_{\mathbb{R}^p}$ and φ, ψ defined in Proposition 67 we have

$$\nabla^2 f(\overline{x}) \ \rho' \ 0 \Longleftrightarrow \left[\left\|H^* u'\right\|^2 \ \rho \ 1 \ \ \forall u' \in S_n\right] \Longleftrightarrow \left[\varphi(u) \ \rho \ 1 \ \ \forall u \in S_n\right],$$

$$\nabla^2 D(\overline{\lambda}) \ \rho' \ 0 \Longleftrightarrow \left[\left\|H v'\right\|^2 \ \rho \ 1 \ \ \forall v' \in S_m\right] \Longleftrightarrow \left[\psi(v) \ \rho \ 1 \ \ \forall v \in S_m\right].$$

From the above considerations we obtain the following result.

Proposition 68 *Let $(\overline{x}, \overline{\lambda}) \in X_0 \times S^-$ be a critical point of Ξ. Consider $E, F \in \mathfrak{M}_n$ such that $E^* E = -A(\overline{\lambda})$, $F^* F = \nabla^2 V(q(\overline{x}))$, $d_i \in \mathbb{R}^n$ $(i \in \overline{1, m})$ and H defined above.*

(i) *If $\overline{x}$ (resp. $\overline{\lambda}$) is a local maximizer of f (resp. D), then $\|Hv\| \leq 1$ for all $v \in S_m$, or, equivalently, $(\alpha =)$ $\beta \leq 1$. Conversely, if $\|Hv\| < 1$ for all $v \in S_m$, then $\overline{x}$ (resp. $\overline{\lambda}$) is a local strict maximizer of f (resp. D). In particular, if $A_i \overline{x} = b_i$ (or equivalently $d_i = 0$) for all $i \in \overline{1, m}$, then $\overline{x}$ and $\overline{\lambda}$ are local strict maximizers of f and D, respectively.*

(ii) *If $\overline{x}$ is a local minimizer of f, then $\|H^* u\| \geq 1$ for all $u \in S_n$; in particular H is surjective, $m \geq n$, and every positive eigenvalue of $H^* \circ H$ is greater than or equal to 1. Conversely, if $\|H^* u\| > 1$ for all $u \in S_n$, then $\overline{x}$ is a local strict minimizer of f; moreover, if $m > n$ then $\overline{\lambda}$ is not a local extremum for D.*

(iii) *If $\overline{\lambda}$ is a local minimizer of D, then $\|Hv\| \geq 1$ for all $v \in S_m$; in particular H is injective, $m \leq n$, and every positive eigenvalue of $H \circ H^*$ is greater than or equal to 1. Moreover, if $m < n$ then $\overline{x}$ is not a local extremum for f. Conversely, if $\|Hv\| > 1$ for all $v \in S_m$, then $\overline{\lambda}$ is a local strict minimizer of D.*

(iv) *Assume that $m = n$ and $\{A_i \overline{x} - b_i \mid i \in \overline{1, m}\}$ is a basis of $\mathbb{R}^m$. If $\|Hv\| > 1$ for all $v \in S_m$, then $\overline{x}$ and $\overline{\lambda}$ are local strict minimizers of f and D, respectively.*

Note that Proposition 68 (iii) gives a positive answer to the question formulated in [42, p. 234]. In [21] one says the C^2 function Π is non-singular if $\nabla \Pi(\overline{x}) = 0 \Rightarrow \det \nabla^2 \Pi(\overline{x}) \neq 0$. Under such a condition we have the following result.

Corollary 69 *Let $(\overline{x}, \overline{\lambda}) \in X_0 \times S^-$ be a critical point of Ξ such that $\det \nabla^2 f(\overline{x}) \neq 0$ [that is 0 is not an eigenvalue of $\nabla^2 f(\overline{x})$]. The following assertions hold:*

(a) *$\overline{x}$ is a local maximizer of f if and only if $\|Hv\| < 1$ for all $v \in S_m$, if and only if $\overline{\lambda}$ is a local maximizer of D.*

(b) *Assume that $m = n$. Then $\overline{x}$ is a local minimizer of f if and only if $\|Hv\| > 1$ for all $v \in S_m$, if and only if $\overline{\lambda}$ is a local minimizer of D.*

Then we analyzed results obtained by DY Gao and his collaborators in papers dedicated to unconstrained optimization problems, related to “triality theorems".

References

[1] J.P. Aubin, H. Frankowska, *Set-Valued Analysis*, Birkäuser, Basel, 1990.

[2] T. Chelmuş, M. Durea, *Exact penalization and optimality conditions for constrained directional Pareto efficiency*, Pure and Applied Functional Analysis, accepted.

[3] T. Chelmuş, M. Durea, E.-A. Florea, *Directional Pareto efficiency: concepts and optimality conditions*, Journal of Optimization Theory and Applications, 182 (2019), 336–365.

[4] R. Cibulka, M. Durea, M. D. Pantiruc, R. Strugariu, *On the stability of the directional regularity*, Set-Valued and Variational Analysis, DOI: 10.1007/s11228-019-00507-2.

[5] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[6] A.L. Dontchev, R.T. Rockafellar, *Implicit functions and solution mappings*, Springer, Berlin, 2009.

[7] M. Durea, *Some cone separation results and applications*, Revue d'Analyse Numerique et de Theorie de l'Approximation, 37 (2008), 37–46.

[8] M. Durea, E.-A. Florea, R. Strugariu, *Henig proper efficiency in vector optimization with variable ordering structure*, Journal of Industrial and Management Optimization, 15 (2019), 791–815.

[9] M. Durea, E.-A. Florea, R. Strugariu, *Efficiencies and optimality conditions in vector optimization with variable ordering structure*, book chapter in Variational Analysis and Set Optimization: Developments and Applications in Decision Making, C. Tammer, A. Khan, E. Köbis (Eds.), CRC Press (Taylor & Francis).

[10] M. Durea, R. Strugariu, *Optimality conditions and a barrier method in optimization with convex geometric constraint*, Optimization Letters, 12 (2018), 923–931.

[11] M. Durea, R. Strugariu, *A barrier method in convex optimization with generalized inequality constraint*, Optimization Letters, DOI: 10.1007/s11590-019-01393-1.

[12] M. Durea, R. Strugariu, *On the sensitivity of Pareto efficiency in set-valued optimization problems*, submitted.

[13] M. Durea, M. Panţiruc, R. Strugariu, *Minimal time function with respect to a set of directions. Basic properties and applications*, Optimization Methods and Software, 31 (2016), 535–561.

[14] M. Durea, M. Panţiruc, R. Strugariu, *A new type of directional regularity for mappings and applications to optimization*, SIAM Journal on Optimization, 27 (2017), 1204–1229.

[15] M. Durea, R. Strugariu, C. Tammer, *On set-valued optimization problems with variable ordering structure*, Journal of Global Optimization, 61 (2015), 745–767.

[16] E.-A. Florea, *Coderivative necessary optimality conditions for sharp and robust efficiencies in vector optimization with variable ordering structure*, Optimization, 65 (2016), 1417–1435.

[17] E.-A. Florea, *Vector optimization problems with generalized functional constraints in variable ordering structure setting*, Journal of Optimization Theory and Applications, 178 (2018), 94–118.

[18] E.-A. Florea, D.-E. Plop (Maxim), *Directional openness for epigraphical mappings and optimality conditions for directional efficiency*, submitted.

[19] DY Gao, V Latorre, N Ruan (eds), *Canonical Duality Theory. Unified Methodology for Multidisciplinary Study*, Advances in Mechanics and Mathematics 37. Springer, Cham, 2017.

[20] DY Gao, N Ruan, V Latorre, *Canonical duality-triality theory: bridge between nonconvex analysis/mechanics and global optimization in complex system*, Canonical duality theory, 1–47, Adv. Mech. Math., 37, Springer, Cham, 2017.

[21] DY Gao, C Wu, *Triality theory for general unconstrained global optimization problems*, in: D Gao, V Latorre, N Ruan (eds) Canonical Duality Theory. Advances in Mechanics and Mathematics 37, pp. 127–153. Springer, Cham (2017).

[22] A. Göpfert, H. Riahi, C. Tammer, C. Zălinescu, *Variational Methods in Partially Ordered Spaces*, Springer, Berlin, 2003.

[23] T.X.D., Ha, *Some variants of the Ekeland variational principle for a set-valued map.* Journal of Optimization Theory and Applications, 124 (2005), 187–206.

[24] A. Hamel, A. Löhne, *Minimal element theorems and Ekeland's principle with set relations*, Journal of Nonlinear and Convex Analysis, 7 (2006), 19–37.

[25] A.H. Hamel, *Variational Principles on Metric and Uniform Spaces*, Habilitation thesis, Martin-Luther University Halle–Wittenberg (2005).

[26] A. Hamel, C. Zălinescu, *A unified approach for EVP type results for set-valued mappings*, submitted.

[27] JB Hiriart-Urruty, *Conditions for global optimality 2*. J. Global Optim. 13, 349–367 (1998).

[28] V Jeyakumar, AM Rubinov, ZY Wu, *Non-convex quadratic minimization problems with quadratic constraints: Global optimality conditions*, Math. Program. 110, 521–541 (2007).

[29] J.B. Lasserre, *On representation of the feasible set in convex optimization*, Optimization Letters, 4 (2010), 1–5.

[30] J.B. Lasserre, *On convex optimization without convex representation*, Optimization Letters, 5 (2011), 549–556.

[31] A.A. Khan, C. Tammer, C. Zălinescu, *Set-valued Optimization*, Springer, Heidelberg (2015).

[32] C.G. Liu, K.F. Ng, *Ekeland's variational principle for set-valued functions*, SIAM Journal on Optimization, 21 (2011), 41–56.

[33] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation, Vol. I: Basic Theory,* Springer, Berlin, 2006.

[34] H.V. Ngai, H.T. Nguyen, M. Théra, *Metric regularity of the sum of multifunctions and applications*, Journal of Optimization Theory and Applications, 160 (2014), 355–390.

[35] J.-P. Penot, *Cooperative behavior of functions, relations and sets*, Mathematical Methods of Operations Research, 48 (1998), 229–246.

[36] J.H. Qiu, *On Ha's version of set-valued Ekeland's variational principle*, Acta Mathematica Sinica, English Series, 28 (2012), 717–726.

[37] J.H. Qiu, *Set-valued quasi-metrics and a general Ekeland's variational principle in vector optimization*, SIAM Journal on Control and Optimization, 51 (2013), 1350–1371.

[38] J.H. Qiu, *A pre-order principle and set-valued Ekeland variational principle*, Journal of Mathematical Analysis and Applications, 419 (2014), 904–937.

[39] Rockafellar RT. Convex Analysis. Princeton (NJ): Princeton University Press; 1972.

[40] R.T. Rockafellar, *Proto-differentiability of set-valued mappings and its applications in optimization*, Annales de l'Institut Henri Poincaré, 6 (1989), 449–482.

[41] C. Tammer, C. Zălinescu, *Vector variational principles for set-valued functions*, Optimization 60 (2011), 839–857.

[42] MD Voisei, C Zalinescu, *Counterexamples to a triality theorem for quadratic-exponential minimization problems*, Math. Meth. Oper. Res. 77 (2013), 227–237.

[43] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, Singapore, 2002.

[44] X.Y. Zheng, K.F. Ng, *The Lagrange multiplier rule for multifunctions in Banach spaces*, SIAM Journal on Optimization, 17 (2006), 1154–1175.

[45] C. Zălinescu, *On quadratic optimization problems and canonical duality theory*, submitted.

[46] C. Zălinescu, *On unconstrained optimization problems solved using the canonical duality and triality theories*, Optimization, accepted.

Project leader,
prof. dr. Constantin Zălinescu