Scientific activity report for the research grant

Variational Analysis over Cones and Applications to Vector Optimization

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The scientific activity in the period from 27.05.2022 (the date of the beginning of the project) to 31.12.2022 (the date of ending of the first stage) was focused on the objectives and activities of the project associated to this stage.

The objective of the stage was entitled *Variational analysis over cones* and the foreseen activities within this objective were

1. Algebraic and topological properties of cones – documentation
2. Algebraic and topological properties of cones – intermediate results
3. Dilating cones and cone separation – documentation

Moreover, the research in this stage was foreseen to result into a Q1 or Q2 scientific paper.

The objective as well as the activities of this stage were completely fulfilled. In our investigations we obtained several results within the proposed objective and these results are contained in the following Q1 or Q2 scientific papers publishes in important mathematical journals (the order takes into account the links with the mentioned objective and activities):

- **M. Durea**, **E.-A. Florea**, D. Maxim, **R. Strugariu**, *Approximate efficiency in set-valued optimization with variable order*, Journal of Nonlinear and Variational Analysis, 6 (2022), 619-640; IF: 1.683 (Q1), RIS: 0.685.
- M. Burlică, **M. Durea**, **R. Strugariu**, *New concepts of directional derivatives for set-valued maps and applications to set optimization*, Optimization, DOI: 10.1080/02331934.2022.2088368; IF: 1.203 (Q1), RIS: 1.124 (Q2).
- **M. Durea**, **R. Strugariu**, *Directional derivatives and subdifferentials for set-valued maps applied to set optimization*, Journal of Global Optimization, DOI: 10.1007/s10898-022-01222-3; IF: 0.977 (Q2); RIS: 1.211 (Q2).

Moreover, some results obtained as well in the framework of this project that are not yet published were presented at *12th International Conference on Parametric Optimization and Related Topics*, (September 12–16, 2022, Augsburg, Germany) by **C. Zălinescu** in the talk entitled *On an open problem related to the parametric version of Gale's example in conic linear programming.*

For more details, see the web page of the project at

http://www.math.uaic.ro/~zalinesc/VARACAVO_en.htm

In what follows, we briefly present the main concepts used and the most important results from the papers listed above.

The paper [2] primarily concerns several notions of conic enlargement (dilating) procedures which are subsequently applied in order to study some conic separation issues and, on this basis, to get optimality conditions for approximate efficiency in various contexts.

Let X be a normed vector space and $C \subset X$ be a closed convex cone. One says that C is based if there is a convex set B such that $0 \notin \operatorname{cl} B$ and $C = \operatorname{cone} B$ (where cone stands for the conic hull). If B is also bounded, one says that C is well-based and in this case, since C is supposed to be closed, one can take B to be closed as well (see [4, Definition 2.2.14]).

We consider and compare four kinds of enlargements for C. Firstly, we note the obvious fact that

$$C = \operatorname{cone}(C \cap S_X)$$

(where S_X is the unut sphere) and we denote by S_C the set $C \cap S_X$. The first three enlargements are defined next.

Definition 1 *Let $\varepsilon > 0$ and $C \subset X$ be a closed convex and pointed cone. Define the following enlargements of the cone C:*

(i) *the first type enlargement is*

$$C^{(1)\varepsilon} = \text{cone}\left(\{x \in S_X \mid d(x, C) \leq \varepsilon\}\right);$$

(ii) *the second type enlargement is*

$$C^{(2)\varepsilon} = \text{cone}\left(\{x \in S_X \mid d(x, S_C) \leq \varepsilon\}\right);$$

(iii) *if C is based with the base B, the third type enlargement is*

$$C^{(3)\varepsilon} = \text{cone}\left(\{x \in X \mid d(x, B) \leq \varepsilon\}\right).$$

To define the fourth type of enlargement, one needs the following auxiliary result.

Lemma 2 *Let $C \subset X$ be a closed, convex cone. Then*

(i) *C is based if and only if there is $x^* \in X^*$ such that $x^*(x) > 0$ for all $x \in C \setminus \{0\}$. In this case $C \cap \{x \in X \mid x^*(x) = 1\}$ is a base for C;*

(ii) *C is well-based if and only if there are $x^* \in X^*$ and $\alpha > 0$ such that $x^*(x) \geq \alpha \|x\|$ for all $x \in C$. In this case $C \cap \{x \in X \mid x^*(x) = 1\}$ is a bounded base for C.*

Definition 3 *Let $C \subset X$ be a closed, convex, well-based cone and $\varepsilon > 0$. Let $x^* \in X^*$ be the functional from Lemma 2 (ii) and $A := \{u \in X \mid x^*(u) = 1\}$. The fourth type enlargement is:*

$$C^{(4)\varepsilon} = \text{cone}\left(\{x \in A \mid d(x, C \cap A) \leq \varepsilon\}\right).$$

Then, we analyze some links between these types of enlargements and we obtain some mutual inclusions, as follows.

Proposition 4 *Let $C \subset X$ be a closed convex cone and let $\varepsilon > 0$. Then*

(i) *$C^{(2)\varepsilon} \subset C^{(1)\varepsilon}$ and there is $\delta > 0$ such that $C^{(1)\delta} \subset C^{(2)\varepsilon}$;*

(ii) *if C is well-based, then there exists $\delta > 0$ such that $C^{(3)\delta} \subset C^{(1)\varepsilon}$ and there exists $\eta > 0$ such that $C^{(1)\eta} \subset C^{(3)\varepsilon}$;*

(iii) *if C is well-based, then there exists $\delta > 0$ such that $C^{(4)\delta} \subset C^{(1)\varepsilon}$ and there exists $\eta > 0$ such that $C^{(1)\eta} \subset C^{(4)\varepsilon}$.*

After that, we establish, for infinite dimensional spaces, some cone separation results in the sense given in [6]. We illustrate this presentation by one of these results.

Proposition 5 *Let X be a reflexive Banach space and P, Q be closed convex cones such that $P \cap Q = \{0\}$. If P is well-based with the base B, then there is U, a weak neighborhood of the origin, such that*

$$\text{cone}\,(B + U) \cap Q = \{0\}.$$

In particular, there are $\varepsilon_i > 0$ such that $P^{(i)\varepsilon_i} \cap Q = \{0\}$, for all $i \in \overline{1,4}$.

On this theoretical foundation, we build an analysis of stability of three types of approximate efficiency defined in the context of vectorial problems with variable order, for sets and also for set-valued maps. More precisely, we give some results that establish that the limit of a sequence of minimal points for perturbations of a set A is a minimal point of A, and, similarly, the limit of a sequence of minimal points for perturbations of a set-valued map F is a minimal point for F. For this, we use the Painlevé-Kuratowski lower and upper limits for a set-valued map F acting between two normed vector spaces X and Y that are defined as follows: for $\overline{x} \in X$,

$$\begin{aligned}\operatorname*{Liminf}_{x \to \overline{x}} F(x) &= \{y \in Y \mid \forall V \in \mathcal{V}(y),\ \exists U \in \mathcal{V}(\overline{x}),\ \forall u \in U,\ F(u) \cap V \neq \emptyset\} \\ &= \{y \in Y \mid \forall x_n \to \overline{x},\ \exists y_n \to y,\ y_n \in F(x_n),\ \forall n \in \mathbb{N}\}\end{aligned}$$

and

$$\operatorname*{Limsup}_{x\to\overline{x}} F(x) = \{y \in Y \mid \forall V \in \mathcal{V}(y), \forall U \in \mathcal{V}(\overline{x}), \exists u \in U, F(u) \cap V \neq \emptyset\}$$
$$= \{y \in Y \mid \exists x_n \to \overline{x}, \exists y_n \to y, y_n \in F(x_n), \forall n \in \mathbb{N}\}.$$

(As usual, $\mathcal{V}(x)$ denotes the system of neighborhoods of x.)

Although in our paper the main setting of the efficiency concept is with respect to variable order, we study as well some possibilities of converting efficiency under variable order into efficiency under fixed order. In fact, some types of efficient elements of a set-valued map F with respect to an enlargement of an ordering cone-valued map $K : X \rightrightarrows Y$ are turned into the proper and approximate minima, respectively, for the same F, with respect to the upper and lower limit of K, respectively.

Proposition 6 *Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ such that there are $U \in \mathcal{V}(\overline{x})$ and $\varepsilon > 0$ such that for all $x \in U$,*
(i) *$K^{(1)\varepsilon}(x)$ is convex and $K^{(1)\varepsilon}(x) \cap (-K(x)) = \{0\}$;*
(ii) *$(F(x) - \overline{y}) \cap \left(-K^{(1)\varepsilon}(x)\right) \subset \{0\}$.*

Take $C = \bigcap_{x\in U} K(x)$ and suppose that there exists $e \in C \setminus \{0\}$. Suppose that there exists $L > 0$ such that for all $x, z \in U$,

$$F(x) \subset F(z) - L\|x - z\| e + K(z).$$

Then, for all $\delta > 0$, there is $U_\delta \in \mathcal{V}(\overline{x})$ such that

$$(F(U_\delta) - \overline{y} + \delta e) \cap \left(-\operatorname*{Liminf}_{x\to\overline{x}} K(x)\right) = \emptyset.$$

We further investigate the properties of the limiting cones and we obtain several result among which we mention the following ones.

Proposition 7 *Suppose that K has well-based closed and pointed cone values in a neighborhood U of $\overline{x}$ and denote by B the set-valued map that associates to every $x \in U$ the corresponding closed bounded base of $K(x)$. If $0 \notin \operatorname{cl} \bigcup_{x\in U} B(x)$ and $\bigcup_{x\in U} B(x)$ is bounded, then $D = \operatorname*{Liminf}_{x\to\overline{x}} B(x)$ is a bounded base for $\operatorname*{Liminf}_{x\to\overline{x}} K(x)$.*

Proposition 8 *Let $K : X \rightrightarrows Y$ be a set-valued map with cone values, $\overline{x} \in X$ and U an open neighborhood of $\overline{x}$. Suppose that there is a cone P with nonempty interior such that*

$$P \subset \bigcap_{x\in U} K(x).$$

If $P + K(x) \subset K(x)$ for all $x \in U$ (in particular, if K has convex values), then the set-valued map $G : X \rightrightarrows Y$,

$$G(x) = \operatorname*{Liminf}_{u\to x} K(u)$$

is lower semicontinuous at every point $x \in U$.

Finally we derive an optimality result for optimization problems with variable order using the conversion to fixed order along with a cone separation result, generalized differentiation calculus and some properties of the well-known Gerstewitz functional.

In the papers [1] and [3] we introduce several concepts of generalized derivatives for set-valued functions that seem to be appropriate to the study of the set-valued optimization problems, that is, problems where the partial order is still governed by cones as in the case of vectorial problems, but instead of working with order between elements we are dealing with set-order relations. This direction of research was inaugurated by Kuroiwa and his collaborators; see [9], [5], [8] for discussion and details.

Let X, Y be normed spaces over the real field $\mathbb{R}$. Consider $K \subset Y$ a closed convex pointed proper cone and let $A, B \subset Y$ be nonempty sets. Define (see [7], for instance) the relations $\preceq_K^l$ and $\preceq_K^u$ by

$$A \preceq_K^l B \iff B \subset A + K$$
$$A \preceq_K^u B \iff A \subset B - K.$$

When K is solid, that is $\operatorname{int} K \neq \emptyset$, one defines also the strict relations $\prec_K^l$ and $\prec_K^u$ by

$$A \prec_K^l B \iff B \subset A + \operatorname{int} K$$
$$A \prec_K^u B \iff A \subset B - \operatorname{int} K.$$

Let $F : X \rightrightarrows Y$ be a set-valued map with nonempty values and $M \subset X$ be a nonempty closed set. The most common way to define, in the literature, the solution for the problem of minimizing F on M with respect to the above orders is given below.

Definition 9 *An element $\overline{x} \in M$ is said to be $l-$minimum for F on M if*

$$\forall x \in M : \ F(x) \preceq_K^l F(\overline{x}) \Longrightarrow F(\overline{x}) \preceq_K^l F(x).$$

A similar definition holds for $\preceq_K^u$. For the local setting, of course, $\overline{x} \in M$ is said to be $l-$local minimum for F on M if there is a neighborhood U of $\overline{x}$ such that

$$\forall x \in M \cap U : \ F(x) \preceq_K^l F(\overline{x}) \Longrightarrow F(\overline{x}) \preceq_K^l F(x).$$

The above concept means that $F(\overline{x}) \subset F(x) + K$ implies that $F(x) \subset F(\overline{x}) + K$, or, in other words, for any $x \in M$ one can have

$$F(\overline{x}) \not\subset F(x) + K \text{ or } F(\overline{x}) \subset F(x) + K \subset F(\overline{x}) + K.$$

Definition 10 *An element $\overline{x} \in M$ is said to be $l-$weak minimum for F on M if*

$$\forall x \in M : \ F(x) \prec_K^l F(\overline{x}) \Longrightarrow F(\overline{x}) \prec_K^l F(x).$$

A similar definition holds for $\prec_K^u$. The definitions in the local setting are obvious.

Consider the following notions whose links with the above defined efficiencies will become clear later.

Definition 11 *Let $F : X \rightrightarrows Y$ be a set-valued map and $\overline{x}, u \in X$. One calls the upper directional derivative of F at $\overline{x}$ in direction u the set, denoted $D^+F(\overline{x})(u)$, of elements $v \in Y$ such that for all $(t_n) \downarrow 0$ and $(u_n) \to u$*

$$\lim_n \frac{e(F(\overline{x}) + t_n v, F(\overline{x} + t_n u_n))}{t_n} = 0. \tag{1}$$

(Here $e(A, B)$ denotes the excess from the set A to the set B.)

Remark 12 *Observe that $D^+F(\overline{x})(u)$ is a closed set and the relation $v \in D^+F(\overline{x})(u)$ is sequentially characterized by: for all $(t_n) \downarrow 0$ and $(u_n) \to u$*

$$\lim_n e\left(\frac{1}{t_n}F(\overline{x}) + v, \frac{1}{t_n}F(\overline{x} + t_n u_n)\right) = 0,$$

and by: for all $(t_n) \downarrow 0$ and $(u_n) \to u$, and for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$F(\overline{x}) + t_n v \subset F(\overline{x} + t_n u_n) + t_n B(0, \varepsilon).$$

($B(0, \varepsilon)$ denotes the open ball of center 0 and radius ε.)

We propose now another concept inspired by the sequential characterization of $D^+F(\overline{x})(u)$.

Definition 13 *Let $F : X \rightrightarrows Y$ be a set-valued map and $\overline{x}, u \in X$. One calls the lower directional derivative of F at $\overline{x}$ in direction u the set, denoted $D^-F(\overline{x})(u)$, of elements $v \in Y$ such that for all $\varepsilon > 0$ there exist $(t_n) \downarrow 0$, $(u_n) \to u$ and $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,*

$$F(\overline{x}) + t_n v \subset F(\overline{x} + t_n u_n) + t_n B(0, \varepsilon).$$

Next we analyze the links of the above concepts with the classical Hadamard directional derivatives. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function and $\operatorname{dom} f := \{x \in X \mid f(x) \in \mathbb{R}\}$ its domain. Take $\overline{x} \in \operatorname{dom} f$ and $u \in X$. We recall that the upper Hadamard directional derivative of f at $\overline{x}$ in the direction u is

$$d_+ f(\overline{x}, u) = \limsup_{t \downarrow 0, u' \to u} \frac{f(\overline{x} + tu') - f(\overline{x})}{t},$$

while the lower Hadamard directional derivative of f at $\overline{x} \in \operatorname{dom} f$ in the direction $u \in X$ is

$$d_- f(\overline{x}, u) = \liminf_{t \downarrow 0, u' \to u} \frac{f(\overline{x} + tu') - f(\overline{x})}{t}.$$

Let $f : X \to Y$ be a function. Then the epigraphical set-valued map with respect to K, $\operatorname{Epi} f : X \rightrightarrows Y$, that is

$$\operatorname{Epi} f(x) = f(x) + K, \ \forall x \in X.$$

Proposition 14 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$, $\overline{x} \in \operatorname{dom} f$, $u \in X$. One has*

(i) $D^-(\operatorname{Epi} f)(\overline{x})(u) = [d_- f(\overline{x}, u), +\infty)$;

(ii) $D^+(\operatorname{Epi} f)(\overline{x})(u) = [d_+ f(\overline{x}, u), +\infty)$;

(iii) $D^- f(\overline{x})(u) \subset [d_- f(\overline{x}, u), d_+ f(\overline{x}, u)]$, *while* $D^+ f(\overline{x})(u) \neq \emptyset$ *forces* $d_- f(\overline{x}, u) = d_+ f(\overline{x}, u)$, *in which case* $D^+ f(\overline{x})(u) = \{d_- f(\overline{x}, u)\}$.

Proposition 15 *Let $f : X \to Y$ be a (Fréchet) differentiable function, and $\overline{x}, u \in X$. Then*

$$\nabla f(\overline{x})(u) + K \subset D^+(\operatorname{Epi} f)(\overline{x})(u).$$

If, in addition, $K + D_Y(0, 1)$ is closed, then

$$D^+(\operatorname{Epi} f)(\overline{x})(u) = \nabla f(\overline{x})(u) + K.$$

For a real-valued function $f : X \to \mathbb{R}$, we consider $K := [0, \infty)$, so $\operatorname{Epi} f = f + [0, \infty)$. For $x \in X$, $\widehat{\partial} f(x)$ denotes the Fréchet subdifferential of f at x, while $\widehat{\partial}^+ f(x) = -\widehat{\partial}(-f)(x)$ denotes the set of upper Fréchet subgradients of f at x. For definitions and details of these well-known generalized differentiation objects, we refer to the first chapter of the monograph [11]. We mention here only that $\widehat{\partial} f(x)$ and $\widehat{\partial}^+ f(x)$ are both nonempty if and only if f is differentiable at x.

Proposition 16 *Let $f : X \to \mathbb{R}$ be a function. Take every $\overline{x}, u \in X$. Then*

$$\left\{ x^*(u) \mid x^* \in \widehat{\partial}^+ f(\overline{x}) \right\} \subset D_{H_-}(\operatorname{Epi} f)(\overline{x})(u).$$

Conversely, if X is finite dimensional and $x^* \in X^*$ satisfies $x^*(u) \in D_{H_-}(\operatorname{Epi} f)(\overline{x})(u)$ for all u, then $x^* \in \widehat{\partial}^+ f(\overline{x})$.

Remark 17 *Proposition 18 and Proposition 16 imply the following necessary optimality condition for nonsmooth optimization problems with geometric constraint:*

$$-\widehat{\partial}^+ f(x) \subset (T_B(M, \overline{x}))^-$$

(where $(T_B(M, \overline{x}))^-$ stands for the negative polar of the Bouligand tangent cone which is denoted by $T_B(M, \overline{x})$), which for X finite dimensional, means

$$-\widehat{\partial}^+ f(x) \subset \widehat{N}(M, \overline{x})$$

(where $\widehat{N}(M, \overline{x})$ stands for the Fréchet normal cone to M at $\overline{x}$).

These generalized derivatives are suitable for getting optimality conditions for set-valued optimization problems, as the next results show. Recall first that for a set $\emptyset \neq A \subset Y$, the set of weakly minimal (similarly for maximal) points is

$$\operatorname{WMin}(A, K) := \{a \in A \mid (A - a) \cap (-\operatorname{int} K) = \emptyset\}.$$

Proposition 18 *Let $\overline{x}$ be a local $l-$weak minimum for F on M and* $\operatorname{WMin}(F(\overline{x}), K) \neq \emptyset$. *Then*

$$D^+ F(\overline{x})(u) \cap -\operatorname{int} K = \emptyset, \ \forall u \in T_B(M, \overline{x}).$$

Proposition 19 *Let X be finite dimensional. Let $\overline{x} \in M$ and suppose that* $\operatorname{WMin}(F(\overline{x}), K) \neq \emptyset$. *If*

$$0 \notin D^{-} \operatorname{Epi} F(\overline{x})(T_B(M, \overline{x}) \setminus \{0\}), \tag{2}$$

then for all $e \in Y$ there is $\mu > 0$ such that $\overline{x}$ is a local $l-$weak minimum for $x \rightrightarrows F(x) - \mu \|x - \overline{x}\| e$ on M.

The following result shows that, under certain conditions, a limit of minimal points is critical (in the generalized sense). For a set-valued map $F : X \rightrightarrows Y$ and $\overline{x} \in X$, one says that F is calm at $\overline{x}$ if there is $L > 0$ and $\varepsilon > 0$ such that for all $x \in B_X(\overline{x}, \varepsilon)$, one has

$$F(x) \subset F(\overline{x}) + L \|x - \overline{x}\| D_Y(0, 1).$$

Proposition 20 *Let $(x_n) \subset X$ be a sequence of $l-$weak minima for F on M. Assume that $x_n \to \overline{x} \in X$,* $\operatorname{WMin}(F(x_n), K) \neq \emptyset$ *for all n, and F is calm at $\overline{x}$. Then*

$$D^{+} F(\overline{x})(u) \cap (-\operatorname{int} K) = \emptyset, \ \forall u \in T_U(M, \overline{x}),$$

where $T_U(M, \overline{x})$ is the tangent cone to M at $\overline{x}$ in the sense of Ursescu.

Moreover, a penalization result in the sense of Clarke can be obtained. Recall that, according to [10, Definition 3.1], a nonempty subset A of Y is said to be $K-$compact if any cover of A with sets of the form $U + K$, with U open, admits a finite subcover.

Proposition 21 *Assume that $\overline{x} \in M$, $F(\overline{x})$ is $K-$compact and $\overline{x}$ is $l-$weak minimum for F on M. Let $e \in K \setminus \{0\}$ and suppose that the following generalized Lipschitz condition holds: there is $L > 0$ such that for all $u \in M$ and $v \in X$*

$$F(u) + L \|u - v\| e \subset F(v) + K.$$

Then $\overline{x}$ is $l-$weak minimum on X (that is, without constraints) for the set-valued map $G : X \rightrightarrows Y$,

$$G(x) = F(x) + L d_M(x) e.$$

Using some calculus rules one gets an optimality result for constrained problems.

Proposition 22 *In the assumptions of Proposition 21, for all $u \in X$ one has*

$$\left(D^{+}(\operatorname{Epi} F)(\overline{x})(u) + L D_{H_-}(\operatorname{Epi}(d(\cdot, M)))(\overline{x})(u) e\right) \cap (-\operatorname{int} K) = \emptyset.$$

In particular,

$$\left(D^{+}(\operatorname{Epi} F)(\overline{x})(u) + (L \|u\| e + K)\right) \cap (-\operatorname{int} K) = \emptyset, \ \forall u \in X$$

and

$$\left(D^{+}(\operatorname{Epi} F)(\overline{x})(u) + [L \|u\|, \infty) e\right) \cap (-\operatorname{int} K) = \emptyset, \ \forall u \in X.$$

Finally, since in the classical case a generalized derivative for a function leads to a concept of subdifferential, we follow the same path to define and study some subdifferentials for set-valued maps.

Definition 23 *Let $F : X \rightrightarrows Y$, and $\overline{x} \in X$. The Fréchet subdifferential of F at $\overline{x}$ is*

$$\widehat{\partial} F(\overline{x}) = \left\{ T \in \mathcal{L}(X, Y) \mid \lim_{x \to \overline{x}} \frac{e(\operatorname{Epi} F(x), \operatorname{Epi} F(\overline{x}) + T(x - \overline{x}))}{\|x - \overline{x}\|} = 0 \right\}. \tag{3}$$

Equivalently, $T \in \widehat{\partial} F(\overline{x})$ iff $T \in \mathcal{L}(X, Y)$ and

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in B(\overline{x}, \delta) : \operatorname{Epi} F(x) \subset \operatorname{Epi} F(\overline{x}) + T(x - \overline{x}) + \varepsilon \|x - \overline{x}\| B(0, 1). \tag{4}$$

Similarly, we can define the upper subdifferential of F at $\overline{x}$ as follows

$$\widehat{\partial}^{+} F(\overline{x}) = \left\{ T \in \mathcal{L}(X, Y) \mid \lim_{x \to \overline{x}} \frac{e(\operatorname{Epi} F(\overline{x}) + T(x - \overline{x}), \operatorname{Epi} F(x))}{\|x - \overline{x}\|} = 0 \right\}. \tag{5}$$

The Hadamard lower subdifferential of F at $\overline{x}$ is

$$\partial_{-} F(\overline{x}) = \{ T \in \mathcal{L}(X, Y) \mid T(u) \in D_{-} \operatorname{Epi} F(\overline{x})(u), \ \forall u \in X \}. \tag{6}$$

We explore the situation of differentiable functions.

Proposition 24 *Let $f : X \to Y$ be a Fréchet differentiable function, and $\overline{x} \in X$.*
(i) *For every $u \in X$, one has that*

$$\nabla f(\overline{x})(u) - K \subset D_{-}(\operatorname{Epi} f)(\overline{x})(u). \tag{7}$$

If, in addition, $K + D(0,1)$ is closed, then

$$D_{-}(\operatorname{Epi} f)(\overline{x})(u) = D_{+}(\operatorname{Epi} f)(\overline{x})(u) = \nabla f(\overline{x})(u) - K \tag{8}$$

and, moreover,

$$\partial_{-} f(\overline{x}) = \{\nabla f(\overline{x})\}.$$

(ii) *One has*

$$\widehat{\partial} f(\overline{x}) = \widehat{\partial}^{+} f(\overline{x}) = \{\nabla f(\overline{x})\}.$$

Proposition 25 *Let $F : X \rightrightarrows Y$ be a multifunction, and $\overline{x} \in X$. Then $\widehat{\partial} F(\overline{x}) \subset \partial_{-} F(\overline{x})$, with equality in case X is finite dimensional.*

The expected and useful generalized Fermat rule holds, as shown below.

Proposition 26 *If $\overline{x}$ is a $l-$weak minimum for $F : X \rightrightarrows \mathbb{R}$ and $F(\overline{x})$ is lower bounded with respect to $[0, +\infty)$, then $0 \in \widehat{\partial} F(\overline{x}) \subset \partial_{-} F(\overline{x})$.*

Trying to get sufficient optimality conditions one arrives at the next result.

Proposition 27 *Suppose that X is finite dimensional,* $\operatorname{WMin}(F(\overline{x}), K) \neq \emptyset$. *If*

$$0 \in \partial_{-}(y^{*} \circ F)(\overline{x}) = \widehat{\partial}(y^{*} \circ F)(\overline{x})$$

for some $y^{*} \in K^{+} \setminus \{0\}$ for which the set $y^{*} \circ F(\overline{x})$ has a minimum, then for any $\theta > 0$, and $e \in \operatorname{int} K$, $\overline{x}$ is a $l-$weak minimum for $x \rightrightarrows F(x) + \theta \|x - \overline{x}\| e$.

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