On zero duality gap and the Farkas lemma for conic programming

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Recently S.A. Clark published an interesting duality result in linear conic programming dealing with a convex cone that is not closed in which the usual (algebraic) dual problem is replaced by a topological dual with the aim to have zero duality gap under certain usual hypotheses met in mathematical finance. We present some examples to show that an extra condition is needed for having the conclusion; this supplementary condition is also provided. We also give counterexamples for three results on hedging prices and simple proofs for two known solvability results (see Propositions 4.1 and 4.2).

Key words: linear conic programming; counterexample; duality; Farkas lemma; hedging prices

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1. Introduction. In is well known that the Farkas lemma outside the polyhedral case can be given in an asymptotic way when closedness conditions are not imposed; see (11) and [10]. Attempts were made to obtain the existence of solutions of the problem $Ax = b$, $x \geq 0$ without using closedness conditions.

A similar problem was considered for the case of duality in linear conic programming. This is not a surprise because such duality results can be obtained by applying a variant of the Farkas lemma. Recently, S. A. Clark [5] obtained a duality theorem for linear conic programming problems without using closedness conditions but in which the algebraic dual is replaced by the so called topological dual. In this note we provide a reformulation of the primal and (topological) dual problems using just a closed linear subspace and its orthogonal, we provide sufficient conditions for having zero duality gap, conditions of the type of those used by Clark, and we show that the result of Clark is not valid even in finite-dimensional spaces. We also show that the non-asymptotic versions of the Farkas lemma obtained by Lasserre [12] and Clark [6] can be deduced easily from the asymptotic versions of this result.

As in [5], we consider $(X, Y)$ a dual pair of real linear spaces with respect to a bilinear form $\langle \cdot, \cdot \rangle$ on $X \times Y$ which separates points. It is known that $X$ and $Y$ become separated locally convex spaces, the topology on $X$ being $\sigma(X,Y)$ determined by the seminorms $x \mapsto |\langle x, y \rangle|$ for $y \in Y$; then the topological dual of $X$ is (identified with) $Y$. Similarly, $Y$ is endowed with the topology $\sigma(Y,X)$ and its topological dual is (identified with) $X$. On $X$ one considers a convex cone (not necessarily closed or pointed) $K$; this means that $\lambda x \in K$ and $x + x' \in K$ for all $\lambda \in \mathbb{R}_+: = [0, \infty)$ and $x, x' \in K$; hence $0 \in K$. The cone $K$ determines as usual a (not necessarily antisymmetric) partial order on $X$ denoted by $\geq$; so, for $x, x' \in X$ we set $x' \geq x$, or equivalently $x \leq x'$, if $x' - x \in K$. For a nonempty set $C \subset X$ we set $C^\circ := \{ y \in Y \mid \langle x, y \rangle \geq 0 \ \forall x \in C \}$, $C^\# := \{ y \in Y \mid \langle x, y \rangle > 0 \ \forall x \in C \setminus \{0\} \}$ and $C^\perp := \{ y \in Y \mid \langle x, y \rangle = 0 \ \forall x \in C \}$; when $C$ is a linear subspace we have that $C^\circ = C^\perp$. In the case $C \subset Y$, $C^\circ$, $C^\#$ and $C^\perp$ are subsets of $X$ and are defined similarly. Of course, $K^o$ is a closed convex cone which induces a partial order on $Y$ denoted also $\geq$ and $(K^o)^o = clK$, where $clC$ or $\overline{C}$ means the closure of $C \subset X$. For $C, D \subset X$, $x \in X$, $\Gamma \subset \mathbb{R}$ and $\gamma \in \mathbb{R}$ we set

\[ C + D := \{ x + x' \mid x \in C, x' \in D \}, \quad x + D := \{ x \} + D, \]
\[ \Gamma C := \{ \gamma x \mid \gamma \in \Gamma, x \in C \}, \quad \gamma C := \{ \gamma \} C. \]

We also denote by $\text{int} C$, $\text{span} C$, $\overline{\text{span}} C$, core $C$ and $\text{icr} C$ the interior, linear hull, closed linear hull, the algebraic interior (or core) and the relative algebraic interior (or intrinsic core) of $C$. One considers also another dual pair $(Z, W)$ whose bilinear form (which separates points) is denoted also by $\langle \cdot, \cdot \rangle$, and a continuous linear operator $A : Y \to W$. Then the adjoint operator $A^* : Z \to X$ defined by $\langle Ax, y \rangle = \langle z, Ay \rangle$ for $z \in Z$ and $y \in Y$. It is well known that $(\text{Im} A^*)^\perp = \ker A$ and $(\ker A)^\perp = cl(\text{Im} A^*)$. The primal problem is then the problem

\[ \sup \langle c, y \rangle \quad \text{s.t.} \quad Ay = b, \quad y \geq 0, \]

and its algebraic dual problem is

\[ \inf \langle z, b \rangle \quad \text{s.t.} \quad A^* z \geq c, \]
where \( b \in W \) and \( c \in X \) are fixed elements. Assuming that the equation \( Ay = b \) has a solution \( \bar{y} \), and setting \( M : = \text{Im}\; A^* \), the problems (1) and (2) become

\[
\sup \langle c, y \rangle \quad \text{s.t.} \quad y \in \bar{y} + M^\perp, \; y \geq 0,
\]

and

\[
\inf \langle x, \bar{y} \rangle \quad \text{s.t.} \quad x \in M, \; x \geq c,
\]

respectively. In order to make smaller the duality gap between these two problems in [5] one replaces \( M \) by \( \text{cl}\; M \) in the last problem, obtaining so the so called \textit{topological dual problem}. So, setting \( L : = \text{cl}\; M = \text{cl}(\text{Im}\; A^*) \) and \( \bar{x} : = c \), the primal problem becomes

\[
\sup \langle \bar{x}, y \rangle \quad \text{s.t.} \quad y \in \bar{y} + L^\perp, \; y \geq 0,
\]

while the topological dual problem becomes

\[
\inf \langle x, \bar{y} \rangle \quad \text{s.t.} \quad x \in L, \; x \geq \bar{x}.
\]

Replacing \( x \) by \( x' \) := \( x - \bar{x} \) in (4), this problem becomes

\[
\inf (\langle x', \bar{y} \rangle + \langle \bar{x}, \bar{y} \rangle) \quad \text{s.t.} \quad x' \in (-\bar{x}) + L, \; x' \geq 0,
\]

which shows that the (sole) difference between problems (3) and (4) is that in (4) the order is not defined by a closed (convex) cone, taking into account that a minimization problem can be transformed easily into a maximization problem. Let us set

\[
\alpha := \sup \{ \langle \bar{x}, y \rangle \mid y \in \bar{y} + L^\perp, \; y \geq 0 \},
\]

\[
\beta := \inf \{ \langle x, \bar{y} \rangle \mid x \in L, \; x \geq \bar{x} \},
\]

where, as usual, \( \sup \emptyset := -\infty \) and \( \inf \emptyset := +\infty \). It is easy to see that \( \alpha \leq \beta \). Indeed, if \( y \in \bar{y} + L^\perp \) with \( y \geq 0 \) and \( x \in L \) with \( x \geq \bar{x} \), then

\[
\langle \bar{x}, y \rangle \leq \langle x, y \rangle = \langle x, y - \bar{y} \rangle + \langle x, \bar{y} \rangle = \langle x, \bar{y} \rangle;
\]

if (3) or (4) is not feasible the inequality \( \alpha \leq \beta \) is obvious.

Observe that for \( L = \{0\} \) one has

\[
\alpha = \begin{cases} 
0 & \text{if } -\bar{x} \in \text{cl}\; K, \\
\infty & \text{if } -\bar{x} \notin \text{cl}\; K
\end{cases}, \quad \beta = \begin{cases} 
0 & \text{if } -\bar{x} \in K, \\
\infty & \text{if } -\bar{x} \notin K
\end{cases}
\]

which confirms that \( \alpha \leq \beta \) and shows that \( \alpha < \beta \) if and only if \( -\bar{x} \in (\text{cl}\; K) \setminus K \). Moreover, if \( L = X \) then \( \alpha = \beta = \langle \bar{x}, \bar{y} \rangle \) for \( \bar{y} \in K^\circ \) and \( \alpha = \beta = -\infty \) otherwise. So we can assume that \( \{0\} \neq L \neq X \) in the sequel.

We note that the pair of dual problems (1) and (2) in this general framework, even in a somewhat more general formulation, appeared in the literature long time ago (see i.e. [9]); conditions for zero duality gap between these problems are given for example in [20] and [18]. In finite-dimensional spaces they are studied mainly in connection with semidefinite programming and are called linear conic (or cone) programming problems; see Nesterov and Nemirovski’s book [14]. The problems (3) and (4) are considered in [14], too. Zero duality gap between these pairs of problems in finite-dimensional spaces is obtained under some interiority conditions we shall mention in the next section.

In the context of mathematical finance one considers frequently the pair of problems (1) and (2) in spaces of measurable functions; in such spaces often the cones under consideration have empty (algebraic) interior and so the interiority conditions can not be envisaged. In [5] the equality \( \alpha = \beta \) is obtained under certain conditions which correspond to some axioms in mathematical finance: no arbitrage (NA), no approximate arbitrage (NAA), no free lunches (NFL). In the next section we show that such (slightly modified) conditions are sufficient to have \( \alpha = \beta \) in finite-dimensional spaces but are not sufficient in infinite-dimensional spaces even if \( K \) is closed; see Example 2.3. In fact, as we shall see below, in finite-dimensional spaces those conditions are equivalent to previously used interiority conditions.
2. Duality results. In the sequel we assume that \( L \) is a proper closed linear subspace of \( X \). Let us consider first some conditions which will be used in the sequel, the framework and notation being that in the preceding section:

- **A.1′**: there exists \( y_0 \geq 0 \) such that \( y_0 \in \overline{y} + L^\perp; \)
- **A.1**: there exists \( y_0 \geq 0 \) such that \( y_0 \in \overline{y} + L^\perp \) and \( y_0 \neq 0; \)
- **A.2′**: there exists \( x_0 \in L \cap K \) such that \( x_0 \neq 0; \)
- **A.2**: there exists \( x_0 \in L \) such that \( \varphi(x_0) > 0 \) for every \( \varphi \in (L \cap K)^\circ \setminus \{0\}. \)

In \( A.2 \) the set \( (L \cap K)^\circ \) is considered as a subset of the topological dual \( L^* \) for the dual pair \( (L, L^*) \).

If \( L \cap K = \{0\} \) then \( (L \cap K)^\circ = L^\ast \), and so \( A.2 \) does not hold; hence \( A.2 \Rightarrow A.2′ \) in this case. In fact, as confirmed by the author in an email communication, in the proof of [5, Th. 3] it is used only \( A.2′ \) even if in its statement it is assumed \( A.2 \). The other conditions are:

- **A.3′**: \( \{ x \in L \cap K \mid \langle x, \overline{y} \rangle = 0 \} \subset -K; \)
- **A.3**: \( \{ x \in L \cap K \mid \langle x, \overline{y} \rangle = 0 \} = \{0\}; \)
- **B.3′**: \( \{ y \in L^\perp \cap K^\circ \mid \langle x, y \rangle = 0 \} \subset -K^\circ. \)

Condition **A.1′** simply says that the primal problem (3) is feasible, or equivalently that \( \overline{y} \in K^\circ + L^\perp \).

Of course, **A.1′** is weaker than **A.1**. The example after Corollary 2.1 shows that **A.1′** is essential for the validity of the conclusion of the duality results. In [5, p. 242] it is mentioned that condition **A.3** is the mathematical formulation of the axiom NAA, which, at its turn, is the natural topological generalization of the axiom NA.

Throughout this paragraph we assume that \( \dim X < \infty \) and \( K \) is closed. As seen in Appendix, if **A.3′** holds then \( \overline{y} \notin (\text{cl}(L^\perp + K^\circ)) \setminus (\text{icr}(L^\perp + K^\circ)). \) Therefore, [**A.1′** and **A.3′**] is equivalent to

\[
\overline{y} \in L^\perp + \text{icr} K^\circ \quad [\text{or } \text{icr}(L^\perp + K^\circ)],
\]

which essentially is a generalized Slater condition. In a similar way, [**A.1′** and **A.3**] is equivalent to

\[
\overline{y} \in \text{int}(L^\perp + K^\circ).
\]

As in [7], we say that the problem (3) is strongly infeasible if \( \overline{y} \notin \text{cl}(L^\perp + K^\circ) \) (or equivalently \( \text{dist}(\overline{y} + L^\perp, K^\circ) > 0 \)) and (3) is weakly infeasible if \( \overline{y} \in (\text{cl}(L^\perp + K^\circ)) \setminus (L^\perp + K^\circ) \). Hence, if **A.3′** holds then \( \overline{y} \) is strongly infeasible or strongly feasible. Of course, **B.3′** is dual to **A.3′**; if **B.3′** holds then \( \overline{y} \notin (\text{cl}(L + K)) \setminus (\text{icr}(L + K)). \)

**Lemma 2.1** Assume that **A.3** holds and \( y \geq 0 \) is such that \( y \in \overline{y} + L \). Then \( \langle x, y \rangle > 0 \) for every \( x \in L \cap K \setminus \{0\}. \)

**Proof.** Indeed, if \( x \in L \cap K \) then \( 0 \leq \langle x, y \rangle = \langle x, y - \overline{y} \rangle + \langle x, \overline{y} \rangle = \langle x, \overline{y} \rangle; \) assuming that \( \langle x, y \rangle = 0 \) we get \( \langle x, \overline{y} \rangle = 0 \), and so \( x = 0 \) by **A.3**. \( \square \)

Hence [**A.1′**, **A.2′** and **A.3**] implies **A.1**.

In the sequel we shall use several times the following implications:

\[
P, Q \subset X \Rightarrow \text{cl}(P + Q) = \text{cl}(P + cl Q) = \text{cl}(cl P + cl Q),
\]

\[
P, Q \subset X, P, Q \text{ convex cones } \Rightarrow (P + Q)^\circ = P^\circ \cap Q^\circ,
\]

\[
P, Q \subset X, P, Q \text{ convex cones } \Rightarrow (\text{cl } P \cap \text{cl } Q)^\circ = \text{cl}(P^\circ + Q^\circ),
\]

\[
S \subset W, S \text{ convex cone } \Rightarrow (A^{-1}(cl S))^\circ = \text{cl}(A^*(S^\circ)).
\]

Of course, if \( P, Q, S \) are linear subspaces instead of being convex cones in the preceding implication \( ^\circ \) can be equivalently replaced by \( \perp \). The implication (7) is valid in any topological vector space (and easy to prove), the implications (8) and (9) are well known (for (8) one uses just the definition, while for (9) one uses a separation theorem), and the implication (10) is stated in [20] for a more general situation. In fact, in the case in which \( S \) is a closed convex cone, (10) asserts

\[
[ Ay \in S \Rightarrow \langle c, y \rangle \geq 0] \Leftrightarrow c \in \text{cl}(A^*(S^\circ)),
\]

(11)
which is the Farkas lemma when $A^*(S^*)$ is closed; this is the case when $\dim W < \infty$ and $S$ is polyhedral, that is, the intersection of a finite number of closed half-spaces.

**Proposition 2.1** Assume that $\alpha' \in \mathbb{R}$ and there exists $y_0 \in \gamma + L^\perp$ such that $y_0 \geq 0$, that is $A.1'$ holds. Then $\alpha' \geq \alpha$ if and only if $\langle \alpha', -\gamma \rangle \in \text{cl} (\mathbb{R}_+ \times K) + \{(x, \gamma), -x \mid x \in L\}$, or equivalently, there exist the nets $(x_i) \subset K$, $(x_i) \subset L$ such that $x_i - x_i' \to \gamma$ and $\limsup \langle x_i, \gamma \rangle \leq \alpha'$.

**Proof.** For the first equivalence one can use [21, Th. 4]. However, for reader’s convenience we give a direct proof. We have

\[
\alpha \leq \alpha' \iff \left[ y \in \gamma + L^\perp, \ y \geq 0 \Rightarrow \langle \gamma, y \rangle \leq \alpha' \right]
\]

\[
\iff [t > 0, \ y \in t\gamma + L^\perp, \ y \geq 0 \Rightarrow \langle \gamma, y \rangle \leq t\alpha']
\]

\[
\iff [t \geq 0, \ y \geq 0, \ y - t\gamma \in L^\perp \Rightarrow t\alpha' - \langle \gamma, y \rangle \geq 0].
\]

To obtain the implication “$\Rightarrow$” in the last equivalence we proceed like follows: take $y \geq 0$, $y \in L^\perp$; then, for $s > 0$, $s y_0 \geq 0$ and $y + s y_0 \in s\gamma + L^\perp$, and so $\langle x, y + s y_0 \rangle = s^{\alpha'}$, whence, for $s \to 0$, we get $\langle x, y \rangle \geq 0$. Consider $B : \mathbb{R} \times Y \to Y$, $B(t, y) := y - t\gamma$ and $\varphi : \mathbb{R} \times Y \to \mathbb{R}$, $\varphi(t, y) := t\alpha' - \langle x, y \rangle$; hence $B^\ast(x) := (\langle x, \gamma \rangle, -x)$. From the preceding equivalences we get

\[
\alpha \leq \alpha' \iff \left[ (t, y) \in \mathbb{R}_+ \times K^0, B(t, y) \in L^\perp \Rightarrow \varphi(t, y) \geq 0 \right]
\]

\[
\iff \varphi \in \text{cl} \left( (\mathbb{R}_+ \times \gamma) \cap B^{-1}(L^\perp) \right)
\]

\[
\iff \varphi \in \text{cl} \left( (\mathbb{R}_+ \times \gamma) \cap B^{-1}(L^\perp) \right) \iff \varphi \in \text{cl} \left( (\mathbb{R}_+ \times K) + B^\ast(L) \right)
\]

\[
\iff (\alpha', -\gamma) \in \text{cl} \left\{ (s + \langle x, \gamma \rangle, x' - x) \mid s \geq 0, x \in L, x' \in K \right\},
\]

whence the conclusion follows immediately. □

As seen in A4) of Appendix, one can prove the preceding proposition using Convex Analysis.

From the preceding result we obtain that $\alpha = \beta$ under a closedness condition.

**Corollary 2.1** Assume that $A.1'$ holds and the set $(\mathbb{R}_+ \times K) + \{(x, \gamma), -x \mid x \in L\}$ is closed. Then $\alpha = \beta$ and $\beta$ is attained when finite.

**Proof.** Because $A.1'$ holds we have that $\beta \geq \alpha > -\infty$; if $\alpha = \infty$ it is nothing to prove. Let $\alpha < \infty$. Then, by Proposition 2.1, $(\alpha, -\gamma) \in (\mathbb{R}_+ \times K) + \{(x, \gamma), -x \mid x \in L\}$, that is, $(\alpha, -\gamma) = (t, t') + ((x, \gamma), -x)$ with $t \geq 0$, $t' \in K$ and $x \in L$. Hence $x = t + t' \geq \gamma$ and so $\beta \leq \langle x, \gamma \rangle = \alpha - t \leq \alpha$. Therefore, $\alpha = \beta$ and $\langle x, \gamma \rangle = \beta$. □

Note that condition $A.1'$ is essential for having the conclusion of the preceding two results. Indeed, take $L$ a proper closed linear subspace of $X$, $K := L$ and $\gamma \in Y \setminus L^\perp$. Then condition $A.1'$ is not satisfied and so $\alpha = -\infty$: moreover $(\mathbb{R}_+ \times K) + \{(x, \gamma), -x \mid x \in L\} = \mathbb{R} \times L$. Hence for $x \in X \setminus L$ the conclusions of Proposition 2.1 and Corollary 2.1 don’t hold.

The next result shows that in finite-dimensional spaces, when the cone $K$ is also closed, we have zero duality gap under quite mild conditions.

**Proposition 2.2** Assume that $A.1'$ and $A.3'$ hold. If $\dim L < \infty$ and $K$ is closed, then $\alpha = \beta$ and $\beta$ is attained when finite.

**Proof.** By Corollary 2.1, it is sufficient to show that $(\mathbb{R}_+ \times K) + S$ is closed, where $S := \{(x, \gamma), -x \mid x \in L\}$. Indeed, $\mathbb{R}_+ \times K$ is a closed convex cone and $S$ is a linear subspace of $\mathbb{R} \times X$ with $\dim S = \dim L < \infty$, and so $S$ is a locally compact closed convex cone. Take $(\gamma, u) \in P := (\mathbb{R}_+ \times K) \cap S$. With $y_0 \in Y$ provided by $A.1'$, we have $u = -x \in K \cap L$ and $0 \leq \gamma = \langle x, y_0 \rangle = -\langle u, y_0 \rangle \leq 0$. It follows that $\gamma = (u, \gamma) = 0$. Since $u \in K \cap L$, from $A.3'$ we get $u \in -K$, and so $\langle \gamma, u \rangle \in (\mathbb{R}_+ \times K) + S$ is closed.

Note that in the case $\dim X < \infty$ another simple proof can be obtained using Convex analysis (see A5) and A7) in Appendix).
Note also that in order to have zero duality gap for problems (1) and (2) when \( K \) is closed (even for a somewhat more general formulation), in [20] (see [20, Th. 5]) one uses a closedness condition introduced in [9]. Such a condition is also used by A. Shapiro (see [18, Prop. 2.6]). In [20, Cor. 11] it is proved that the closedness condition is satisfied if the interiority condition \((y_0 + \text{int } Q) \cap A(P) \neq \emptyset\) holds (see [20, (6.10)]); note that this interiority condition is equivalent to condition [18, (2.20)] when \( \text{int } Q \) is nonempty. In the context of problems (1) and (2) the interiority condition [18, (2.20)] reads as

\[-c \in \text{int}(K + \text{Im } A^*).\]

However, such a condition can be found earlier in the context of linear conic and semidefinite programming problems in finite-dimensional spaces (see [14] and the recent paper [13], more precisely [13, Th. 2.1]). In this paragraph we assume that \( X = Y = \mathbb{R}^n \) and \( K \) is a closed convex cone. As mentioned in [15], de Klerk, Roos and Terlaky [11] established the strong duality of problems (1) and (2) under the more general condition \((\mathcal{g} + \text{Im } A^*) \cap \text{icr } K^o \neq \emptyset\); this condition is equivalent to

\[\mathcal{g} \in \text{icr}(K^0 + \text{Im } A^*).\]

As mentioned in Introduction, the problems (3) and (4) were considered by Nesterov and Nemirovski in [14] and the conclusion of Proposition 2.2 was obtained under the condition

\[[\mathcal{g} - \text{int } K^o] \cap L^+ \neq \emptyset \tag{12}\]

and \((-\mathcal{P} + L) \cap K \neq \emptyset\). Luo, Sturm and Zhang in [7, Th. 3] obtained the conclusion of Proposition 2.2 under condition (12); it is clear that condition (12) is stronger than condition (6), which, at its turn, is stronger than condition (5). As observed above, \([A1', A3']\) is equivalent to condition (5); hence Proposition 2.2 and [15, Th. 2.2] are equivalent.

If \( K \) is not closed in Proposition 2.2, its conclusion could be false (in the sense that \( \alpha < \beta \)) even if \( A1, A2 \) and \( A3 \) hold, as the next example shows. The main result of [5] is Theorem 3.

Theorem 3 in [5] (with our notations) asserts: Suppose A1, A2, and A3 hold. Then, the optimal value of the primal problem (3) is equal to the optimal value of its topological dual (4).

**Example 2.1** Let \( X = Y = \mathbb{R}^3 \), \( L := \{0\} \times \{0\} \times \mathbb{R} \) (hence \( L^+ = \mathbb{R} \times \mathbb{R} \times \{0\} \)) and \( K := \mathbb{R}_+ \times (\{0, 0, 0\} \cup \text{int } P) \) with \( P := (\mathbb{R}_+)^3 \). Take \( \mathcal{P} := (-1, 0, 0) \) and \( \mathcal{g} := (0, 0, 1) \). Then \( L \cap K = \mathbb{R}_+ \times (0, 0, 1) \), and so \( A2 \) and \( A3 \) hold. Moreover \( K^o = P \), and so \( \mathcal{g} \in K^o \setminus \{0\} \) (hence \( A1 \) holds), \( \{y \in \mathcal{g} + L^+ \mid y \geq 0\} = \mathbb{R}_+ \times \mathbb{R}_+ \times \{1\} \) and \( \{x \in L \mid x \geq \mathcal{P}\} = \emptyset \). It follows that \( \alpha = 0 \) and \( \beta = \infty \).

Another example with \( K \) not closed in which the conditions \( A1, A2\) and \( A3 \) hold and \( \alpha < \beta \) is the next one.

**Example 2.2** Let \( X = Y = \mathbb{R}^4 \), \( L := \mathbb{R} \times \{0\} \times \{0\} \times \mathbb{R} \) (hence \( L^+ = \{0\} \times \mathbb{R} \times \mathbb{R} \times \{0\} \)), \( K := \mathbb{R}_+ \times (\{0, 0\} \cup \text{int } P) \) with \( P := \{(u, v, w) \in \mathbb{R}^3 \mid u, w \geq 0, u^2 \leq 2vw\} \) (hence \( \text{int } P = \{(u, v, w) \mid v, w > 0, u^2 < 2vw\} \)). Take \( \mathcal{P} := (0, 1, 0, 0) \) and \( \mathcal{g} := (1, 0, 0, 0) \). Then \( L \cap K = \{(s, 0, 0, 0) \mid s \geq 0\} \), and so \( \{x \in L \mid (x, \mathcal{g}) = 0\} = \emptyset \); hence \( A2 \) and \( A3 \) hold. Moreover \( K^o = \mathbb{R}_+ \times P \), and so \( \mathcal{g} \in K^o \setminus \{0\} \) (showing that \( A1 \) holds), \( \{y \in \mathcal{g} + L^+ \mid y \geq 0\} = \{1\} \times \{0\} \times \mathbb{R}_+ \times \{0\} \) and \( \{x \in L \mid x \geq \mathcal{P}\} = \emptyset \). It follows that \( \alpha = 0 \) and \( \beta = \infty \).

As we have seen in Proposition 2.2, when \( L \) is finite-dimensional and \( K \) is closed, we can ensure that \( \alpha = \beta \) under much weaker conditions than \( A1, A2 \) and \( A3 \) (it is sufficient the conditions \( A1' \) and \( A3' \) be satisfied). The next proposition shows that when \( L \) is infinite-dimensional and \( K \) is closed conditions \( A1, A2 \) and \( A3 \) are not sufficient to have \( \alpha = \beta \). In this example all the sequences are indexed by \( n \in \mathbb{N} \) := \{1, 2, \ldots\}.

**Example 2.3** Consider an infinite-dimensional real Hilbert space (i.e. \( X = \mathcal{L} \)) with the orthonormal basis \((\alpha_n)n \geq 1\) and \( \eta_n, \mu_n \in (0, 1) \) with \( \eta_n^2 + \mu_n^2 = 1 \) for every \( n \geq 1 \) and \( (\eta_n) \in \mathcal{L} \). Consider, similarly to [3, Exer. 39, p. 80], \( \zeta_n := \eta_n e_{2n} + \mu_n e_{2n-1} \) and \( \zeta'_n := \eta_n e_{2n-1} - \mu_n e_{2n} \) for \( n \geq 1 \). Note that \( (\eta_n, \lambda_n) = \)}
\[ \langle x', y' \rangle = \delta_{nm} (\delta_{nm} \text{ being the Kronecker's symbols}) \] for \( n, m \geq 1 \). Consider \( L := \text{span} \{ e_{2n-1} \mid n \geq 1 \} \), \( L_1 := \text{span} \{ z_n \mid n \geq 1 \} \), and so
\[
L = \left\{ \sum_{n \geq 1} \lambda_n e_{2n-1} \mid (\lambda_n) \in \ell_2 \right\}, \quad L_1 = \left\{ \sum_{n \geq 1} \lambda_n z_n \mid (\lambda_n) \in \ell_2 \right\},
\]
and \( L \cap L_1 = \{0\} \). It is clear that \( \{e_1, e_2, \ldots, e_{2n}\} \subset \text{span} \{e_1, e_3, \ldots, e_{2n-1}, z_1, \ldots, z_n\} \subset L + L_1 \), whence \( X = \text{cl}(L + L_1) \). Moreover, \( \pi := -\sum_{n \geq 1} \eta_n e_{2n} \notin L + L_1 \). We have that
\[
L^\perp = \left\{ \sum_{n \geq 1} \lambda_n e_{2n} \mid (\lambda_n) \in \ell_2^* \right\}, \quad L_1^\perp = \left\{ \sum_{n \geq 1} \lambda_n z'_{n} \mid (\lambda_n) \in \ell_2 \right\}.
\]
Consider also \( P := \left\{ \sum_{n \geq 1} \lambda_n e_{2n-1} \mid (\lambda_n) \in \ell_2^+ \right\} \subset L \) and \( K := \text{cl}(P + L_1) \); so \( K \) is a closed convex cone. We take \( Y := X \), the pairing between \( X \) and \( Y \) being given by the scalar product; then
\[
K^\circ = (P + L_1)^\circ = P^\circ \cap L_1^\perp = (P + \text{span} \{ e_{2n} \mid n \geq 1 \}) \cap \text{span} \{ z'_n \mid n \geq 1 \}
\]
\[
= \left\{ \sum_{n \geq 1} \gamma_n e_{2n} \mid (\gamma_n) \in \ell_2, \gamma_{2n-1} \eta_n \geq \gamma_{2n} \mu_n \forall n \geq 1 \right\}.
\]
We have that \( L \cap K = P \). The inclusion \( \supset \) being obvious, let \( x := \sum_{n \geq 1} \lambda_n e_{2n-1} \in K \). From (13) we obtain that \( \lambda_n \eta_n \geq 0 \), and so \( \lambda_n \geq 0 \); hence \( x \in P \). So, for \( L \cap K = P \) seen as a convex cone in \( L \), we have that \( (L \cap K)^\circ = P \) and \( ((L \cap K)^\circ)^\# = \left\{ \sum_{n \geq 1} \lambda_n e_{2n-1} \mid (\lambda_n) \in \ell_2^\# \right\} \neq 0 \). Hence condition A.2 is satisfied. Consider \( \bar{y} := \sum_{n \geq 1} \bar{\lambda}_n z'_n \) with \( (\bar{\lambda}_n) \in \ell_2^\# \). Then \( \bar{y} \in K^\circ \) and \( \{x \in L \cap K \mid \langle x, \bar{y} \rangle = 0\} = \{0\} \). These show that A.1 and A.3 are satisfied, too. Moreover, \( \{y \in K^\circ \mid y \in \bar{y} + L^\perp\} = \{\bar{y}\} \). Indeed, let \( y := \sum_{n \geq 1} \lambda_n z'_n \) with \( (\lambda_n) \in \ell_2^* \) such that \( y - \bar{y} \in L^\perp \). Then \( \sum_{n \geq 1} (\lambda_n - \bar{\lambda}_n) \eta_n e_{2n-1} - \sum_{n \geq 1} (\lambda_n - \bar{\lambda}_n) \mu_n e_{2n} = \sum_{n \geq 1} \lambda_n' e_{2n} \) with \( (\lambda_n') \in \ell_2 \). It follows that \( (\lambda_n - \bar{\lambda}_n) \eta_n \geq 0 \) for \( n \geq 1 \), whence \( y = \bar{y} \). Therefore, \( \alpha := \sup \{ \langle x, y \rangle \mid y \in \bar{y} + L^\perp, y \geq 0 \} = \langle \bar{\pi}, \bar{y} \rangle \). On the other hand, assume that \( x := \sum_{n \geq 1} \lambda_n e_{2n-1} \in L \) (with \( (\lambda_n) \in \ell_2 \)) is such that \( x \geq \bar{\pi} \), that is, \( \sum_{n \geq 1} \lambda_n e_{2n-1} + \sum_{n \geq 1} \eta_n e_{2n} \in K \). Then \( \lambda_n \eta_n \geq \lambda_n \mu_n \), that is, \( \lambda_n \geq \mu_n \) for every \( n \geq 1 \). This is a contradiction because \( \mu_n \to 1 \). Therefore, \( \beta = \inf \{ \langle x, y \rangle \mid x \in L, x \geq \bar{\pi} \} = \infty \).

Note that Examples 2.1 and 2.3 are effectively counterexamples for [5, Th. 3] because in a Hilbert space \( X \) every closed linear subspace \( L \) is the image of a continuous linear operator; more precisely, \( L \) is the image of the orthogonal projection onto \( L \). In fact, in the proof of [5, Th. 3] condition A.2 was used instead of A.2. So a natural question is what is needed to add besides the conditions A.1’ and A.3’ in Proposition 2.2 in order to have \( \alpha = \beta \) without the closedness of \( K \) even if \( \dim X < \infty \). As seen in Example 2.2 adding A.2’ (or even A.2 as seen in Example 2.1) is not sufficient. The next result provides such conditions; in its proof we follow the lines of the proof of [5, Th. 3].

**PROPOSITION 2.3** **Assume that A.1’, A.2’ and A.3 hold.** If \( \dim X < \infty \) and B.3’ holds, then \( \alpha = \beta \).

**PROOF.** Fix \( y_0 \in \bar{y} + L^\perp \) with \( y_0 \geq 0 \) and \( x_0 \in L \cap K \setminus \{0\} \). Set \( F := \{ x \in L \mid \langle x, \bar{y} \rangle = 0 \} \); by A.3 we have that \( K \cap F = \{0\} \). It is clear that \( \langle \bar{x}, y_0 \rangle \leq \alpha \leq \beta \). If \( \beta = \langle \bar{x}, y_0 \rangle \) then \( \alpha = \beta \). Assume that \( \beta > \langle \bar{x}, y_0 \rangle \). Then \( \bar{x} \notin L \); otherwise \( \langle \bar{x}, y_0 \rangle = \langle \bar{x}, y_0 \rangle \leq \langle x_0, y_0 \rangle = \langle x, y \rangle \) for all \( x \in L \) with \( x \geq \bar{x} \), whence \( \beta = \langle \bar{x}, y_0 \rangle \). Take \( \langle \bar{x}, y_0 \rangle < \lambda \leq \beta \). Consider \( M := L + \mathbb{R} \bar{x} \) and define \( \psi : M \to \mathbb{R} \) by \( \psi(x + t\bar{x}) := \langle x, \bar{y} \rangle + t \langle \bar{x}, y_0 \rangle < \langle x, \bar{y} \rangle + t \lambda = \psi(u) \). Set \( G := \ker \psi = \{ u \in M \mid \psi(u) = 0 \} \). It follows that \( K \cap G = \{0\} \). Since \( x_0 \in L \cap K \setminus \{0\} \), by Lemma 2.1 we get \( \langle x_0, y_0 \rangle > 0 \), and so \( 0 \notin \ker K \) (otherwise \( K \) is a linear subspace, and so, because \( y_0 \in K^\circ \), we must have \( \langle x, y_0 \rangle = 0 \) for every \( x \in K \)). Hence \( \ker G \cap \ker K = G \cap \ker K = \{0\} \). This shows that we can separate \( K \) and \( G \) (in the space \( \text{span}(G - K) = G + K - K \)), that is, there exists \( y \in Y \) which is not null on \( \text{span}(G - K) \) such that \( \langle x', y \rangle \geq \langle x, y \rangle \) for all \( x' \in K \) and \( x \in G \). It follows that \( y \in K^\circ \) and \( \langle x, y \rangle = 0 \) for every \( x \in G = \ker \psi \). Hence there exists \( \mu \in \mathbb{R} \) such that \( \langle x, y \rangle = \mu \psi(x) \) for every
Assume that \( \mu = 0 \). Then \( \langle x, y \rangle = 0 \) for every \( x \in L \), that is, \( y \in L^\perp \), and \( \langle \tau, y \rangle = 0 \). By B.3' we obtain that \( y \in -K^\circ \). It follows that \( \langle x, y \rangle = 0 \) for every \( x \in K \), which implies that \( \langle x, y \rangle = 0 \) for every \( x \in \text{span}(G - K) \), a contradiction. Therefore, \( \mu \neq 0 \). Since \( 0 \leq \langle x_0, y \rangle = \mu \langle x_0, \overline{\tau} \rangle \) we get \( \mu > 0 \), and so we can take \( \mu = 1 \) (replacing \( y \) by \( \mu^{-1}y \) if necessary). Hence \( \langle x + t\overline{\tau}, y \rangle = \langle x, \overline{\tau} \rangle + t\lambda \) for \( x \in L \) and \( t \in \mathbb{R} \), whence \( y - \overline{\tau} \in L^\perp \) and \( \langle \tau, y \rangle = \lambda \). This shows that \( \alpha \geq \lambda \). Since \( \lambda \in \langle (\tau, y_0), \beta \rangle \) is arbitrary, we obtain that \( \alpha \geq \beta \). The proof is complete. \( \square \)

In A6) of Appendix it is given a proof of Proposition 2.3 (provided by one of the referees) which does not use conditions A.2' and A.3. Note that \( L^\perp \cap K^\circ = \{0\} \) in Example 2.3, and so condition B.3' holds.

Therefore, even if \( K \) is closed, the infinite-dimensional version Proposition 2.3 might be not true.

An inspection of the proof of Proposition 2.3 shows that we used the fact that \( \text{dim} X < \infty \) for separating the sets \( G \) and \( K \) by a closed hyperplane which does not contain \( \text{span}(G-K) \). In fact, if \( L \) has codimension 1 (and so \( M = X \)), there is no need to use a separation theorem because \( \psi \) does the job.

A situation when the separation is possible is when the interior of \( G - K \) is nonempty for a compatible topology, in particular when the interior of \( K \) is nonempty for such a topology. In fact, under conditions A.1', A.2' and A.3, we find the desired \( y \) if and only if \( x_0 \notin \text{cl}(G - K) \). Indeed, if \( x_0 \notin \text{cl}(G - K) \) then there exists \( y \in Y \setminus \{0\} \) such that \( \langle x - x', y \rangle \leq 0 < \langle x_0, y \rangle \) for all \( x \in G \) and \( x' \in K \). It follows that \( y \in K^\circ \setminus \{0\} \) and \( \langle x, y \rangle = 0 \) for every \( x \in G = \text{ker} \psi \). Hence there exists \( \mu \in \mathbb{R} \) with \( \langle x, y \rangle = \mu \psi(x) \) for every \( x \in M = L + \mathbb{R} \mathbb{R} \). In particular, \( 0 < \langle x_0, y \rangle = \mu \langle x_0, \overline{\tau} \rangle \), and so \( \mu > 0 \); hence we can (and do) assume \( \mu = 1 \). From \( \langle x, y \rangle = \psi(x) \) for \( x \in M \) we get \( y - \overline{\tau} \in L^\perp \) and \( \langle \tau, y \rangle = \lambda \). Hence \( \alpha \geq \lambda \).

Conversely, assuming the existence of \( y \in K^\circ \) with \( y - \overline{\tau} \in L^\perp \) and \( \langle \tau, y \rangle = \lambda \) we obtain that \( \langle x, y \rangle = \psi(x) \) for \( x \in M \) (and \( \langle x_0, y \rangle = \langle x_0, \overline{\tau} \rangle > 0 \)). Hence, for \( x' \in K \) and \( u \in G \) we get \( \langle u - x', y \rangle = \psi(u) - \langle x', y \rangle \leq 0 \), and so \( x_0 \notin \text{cl}(G - K) \).

Another situation in which the conclusion of Proposition 2.3 holds with \( \text{dim} X = \infty \) is when every positive linear functional on \( X \) is continuous and \( \text{icr} K \) (or more generally \( \text{icr}(K + L) \)) is nonempty. For example, every positive linear functional on \( X \) is continuous when \( Y \) is the algebraic dual of \( X \) or when \( X \) is a Banach lattice.

Let us follow the proof of Proposition 2.3 in the case of Example 2.1 to realize where was the drawback in the proof of [5, Th. 3]. We take \( y_0 := \overline{\tau} \in (\overline{\tau} + L^\perp) \cap K^\circ \). Then \( F = \{x \in L \mid \langle x, \overline{\tau} \rangle = 0\} = \{(0, 0, 0)\} \); clearly \( K \cap F = \{0\} \). Moreover, consider \( 0 = \langle \tau, y_0 \rangle < \lambda < \beta = \infty \). Then \( M := L + \mathbb{R} \mathbb{R} = \mathbb{R} \times \{0\} \times \mathbb{R} \) and \( \psi(t, 0, w) = w - t\lambda \) for \( t, w \in \mathbb{R} \). Hence \( G := \text{ker} \psi = \{t(0, t\lambda) \mid t \in \mathbb{R} \} \), and so we have the confirmation that \( K \cap G = \{0\} \). Moreover,

\[
G - K \subset G - \text{cl} K = \{(t - u, -v, t\lambda - w) \mid t \in \mathbb{R}, u, v, w \geq 0\} = \mathbb{R} \times \mathbb{R}_- \times \mathbb{R},
\]

and so \( \{0\} \neq K \cap L = \{0\} \times \{0\} \times \mathbb{R}_+ \subset \text{cl}(G - K) = \mathbb{R} \times \mathbb{R}_- \times \mathbb{R} \neq X \).

It is possible to give examples in which conditions A.1, A.2', A.3 are satisfied and to have that \( \text{cl}(G - K) = X \).

3. On some results about hedging prices. In [4, Lem. 5] one considers a closed linear subspace \( L \) of a separated locally convex space \( X \), a continuous linear functional \( \pi : L \to \mathbb{R} \) and a convex cone \( C \subset X \) such that \( C \cap L \neq \emptyset \) and \( C \cap F = \emptyset \), where \( F := \{x \in L \mid \pi(x) \leq 0\} \); the condition \( C \cap F = \emptyset \) is denoted by \( \text{NAA} \) in [4] and below. In fact, in the context of [4], \( L \) is \( \overline{\text{M}} \) and \( \pi \) is \( \overline{\text{F}} \). One associates the so called upper and lower hedging prices \( \pi_u(x) \) and \( \pi_l(x) \) to any \( x \in X \) by

\[
\pi_u(x) := \inf \{\pi(x') \mid x' \in L, x' \geq x\}, \quad \pi_l(x) := \sup \{\pi(x'') \mid x'' \in L, x'' \leq x\},
\]

where \( x' \geq x \) and \( x \leq x' \) mean \( x'-x \in C_0 := C \cup \{0\} \). It is clear that \( \pi_l(x) \leq \pi_u(x) \) for every \( x \in X \) (because \( \pi(x'') \leq \pi(x') \) for \( x', x'' \in L \) with \( x'' \leq x' \)) and \( \pi_l(x) = \pi_u(x) = \pi(x) \) for every \( x \in L \). In fact \( \pi_u \) is a sublinear functional with values in \( \mathbb{R} \) (that is, \( \pi_u(0) = 0 \), \( \pi_u(tx) = t\pi_u(x) \) and \( \pi_u(x + x') \leq \pi_u(x) + \pi_u(x') \) for all \( x, x' \in X \) and \( t > 0 \) with the convention \( +\infty + (-\infty) := +\infty \)), and \( \pi_l(-x) = -\pi_u(x) \) for every \( x \in X \). Moreover, setting

\[
\mathcal{P} := \{\varphi \in X^* \mid \forall x \in C, \varphi(x) \geq 0\} \subset X^*, \quad \forall x \in C\},
\]

\( X^* \) being the topological dual of \( X \), for \( \varphi \in \mathcal{P} \) one has \( \pi_l(x) \leq \varphi(x) \leq \pi_u(x) \) for every \( x \in X \); in particular

\[
\mathcal{P} \neq \emptyset \implies [\pi_l(x) < +\infty, \pi_u(x) > -\infty \forall x \in X].
\]
Lemma 5 in [4] asserts: Suppose every positive linear functional on $X$ is continuous. If NAA holds, then the following conditions are pairwise mutually equivalent: (i) $\pi_u(x) > -\infty$ for every $x \in X$; (ii) $\pi_t(x) < +\infty$ for every $x \in X$; (iii) $\mathcal{P} \neq \emptyset$.

The implication (14) proves that (iii) implies (i) and (ii) in [4, Lem. 5]. For obtaining the implication (i) $\Rightarrow$ (iii) in [4] the author says “Although the algebraic Hahn–Banach theorem usually presumes $f$ is a real-valued sublinear functional, its standard proof remains valid when $f$ is also allowed to take the value $+\infty$”. As seen in [19] and [1], the Hahn–Banach extension theorem for extended valued sublinear functionals is not true. This means that the proof of [4, Lem. 5] is not correct. Below we provide a counterexample for the implication (i) $\Rightarrow$ (iii) in [4, Lem. 5].

Example 3.1 Consider $E$ an infinite-dimensional linear space and $p : E \to \mathbb{R} \cup \{\infty\}$ a sublinear functional which is not minorized by any linear functional (see [19] and [1] for such examples). Let $X := E \times \mathbb{R}$, $L := \{0\} \times \mathbb{R}$, $C := \text{epi} \ p := \{(x, t) \in E \times \mathbb{R} \mid p(x) < t\}$ and $\pi : L \to \mathbb{R}$ be defined by $\pi(0, t) := t$. Taking on $X$ the locally convex topology determined by all the seminorms on $X$, we have that $L$ is a closed linear subspace, $\pi$ is continuous and any linear functional on $X$ is continuous. Moreover, $(0, 1) \in L \cap C$ and $C \cap F = \emptyset$. For $(x, s) \in X \setminus L$ (that is, $x \neq 0$), we have

$$\pi_u(x, s) = \inf \{t \in \mathbb{R} \mid (0, t) - (x, s) \in C_0\} = \inf \{t \in \mathbb{R} \mid p(-x) < t - s\} = s + p(-x),$$

and so $\pi_u(x, s) > -\infty$ for all $(x, s) \in X$. Assume that there exists some $x \in \mathcal{P}$. Then $\varphi(x, t) = \theta(x) + t\alpha$ for some linear functional $\theta : E \to \mathbb{R}$ and $\alpha \in \mathbb{R}$. Because $\varphi |_L = \pi$ and $\varphi(x, t) \geq 0$ for all $(x, t) \in \text{epi} \ p$, we obtain that $\alpha = 1$ and $p(x) < t \Rightarrow \theta(x) + t \geq 0$, that is, $p(x) \geq -\theta(x)$ for every $x \in E$. The last assertion contradicts the choice of $p$. Hence $\mathcal{P} = \emptyset$.

In this situation it is natural to ask about sufficient conditions for having the conclusion of [4, Lem. 5]. In fact $\mathcal{P} = \partial \pi_u(0)$, the subdifferential being taken in the sense of convex analysis (see f.i. [22]). But a necessary and sufficient condition for the nonemptiness of the subdifferential at 0 of a proper sublinear functional is its lower semicontinuity at 0 (see [1, Th. (1.8)], [22, Th. 2.4.14]). Of course, because $\partial \pi_u(x) \subset \partial \pi_u(0)$ for every $x \in \text{dom} \ \pi_u = L - C_0$, one has that $\partial \pi_u(0) \neq \emptyset$ if $\partial \pi_u(x) \neq \emptyset$ for some $x \in \text{dom} \ \pi_u$. In the algebraic case (taking on $X$ the locally convex topology determined by all the seminorms on $X$; then $X^* = X'$), one has $\partial \pi_u(x) \neq \emptyset$ at any $x \in \text{icr} \ (\text{dom} \ \pi_u) = \text{icr} \ (L - C_0)$ provided $\pi_u$ does not take the value $-\infty$. In particular, [4, Lem. 5] holds if $\text{icr} \ C \neq \emptyset$. Hence [4, Lem. 5] holds in finite-dimensional spaces without supplementary conditions; in the infinite-dimensional setting a sufficient condition, as mentioned above, is $\text{icr} \ (L - C_0) \neq \emptyset$.

The other two results on hedging prices in [4] we are concerned are Lemma 6 and Theorem 7.

Lemma 6 in [4] asserts: Suppose $\mathcal{P} \neq \emptyset$ and NAA holds. Then for every $x \notin L$ and $\lambda \in \mathbb{R}$ such that $\pi_u(x) > \lambda > \pi_t(x)$, there exists some $p \in \mathcal{P}$ for which $p(x) = \lambda$.

Theorem 7 in [4] asserts: Suppose $\mathcal{P} \neq \emptyset$ and NAA holds. Then the following conditions on a contingent claim $x \in X$ are pairwise mutually equivalent: (i) $x \in \mathcal{M}$; (ii) $x$ is priced by arbitrage; (iii) $\pi_u(x) = \pi_t(x)$.

Because we refer only to (ii) and (iii) in [4, Th. 7] we just recall that $x \in X$ is priced by arbitrage (see [4, p. 170]) if $\{p(x) \mid p \in \mathcal{P}\}$ is a singleton. The framework of Example 2.2 can be used to give counterexamples to the results cited above.

Example 3.2 With the notation from Example 2.2, take $C := K \setminus \{0\}$ and $\pi : L \to \mathbb{R}$ defined by $\pi(x) := \langle x, y \rangle$. Then $C \cap L \neq \emptyset$, $\gamma$ (seen as a linear function defined on $X$) is in $\mathcal{P}$, $F = \mathbb{R} \times \{0\} \times \{0\} \times \mathbb{R}$ is closed and $C \cap F = \emptyset$, that is, condition NAA holds. Moreover, $\pi \notin \mathcal{P}$ and $\{x' \in L \mid x'' \leq \pi\} = \{x'' \in L \mid x'' \leq \pi\} = \emptyset$, whence $\pi_u(\pi) = \infty$ and $\pi_t(\pi) = -\infty$. A simple calculation shows that $\mathcal{P} = \{y \in \mathbb{R} + L^+ \mid y \geq 0\} = \{1\} \times \{0\} \times \mathbb{R}_+ \times \{0\}$, and so $p(\pi) = 0$ for every $p \in \mathcal{P}$, contradicting [4, Lem. 6]. Because $\pi_t(\pi) < \pi_u(\pi)$, we have that the implication (ii) $\Rightarrow$ (iii) of [4, Th. 7] is not true.

4. On some non-asymptotic Farkas’ lemma type results. We use in this section the same framework and notation as in Introduction.

The problem of the solvability (feasibility) of the equation $Ay = b$ with $y \geq 0$ is an important one, and is related to the classic Farkas’ lemma. Of course, the equation $Ay = b$ with $y \geq 0$ is feasible if and only
if \( b \in A(K^\circ) \). On the other hand, we have that \( A^*z \geq 0 \Rightarrow \langle z, b \rangle \geq 0 \) if and only if \( b \in ((A^*)^{-1}(K))^\circ \). But, by (10),
\[
((A^*)^{-1}(clK))^\circ = cl\{A(K^\circ)\}.
\]
(15)
So, there is no chance to obtain that
\[
[A^*z \geq 0 \Rightarrow \langle z, b \rangle \geq 0] \Leftrightarrow \exists y \in K^\circ : Ay = b
\]
(16)
if \((A^*)^{-1}(clK) \neq cl\{(A^*)^{-1}(K)\} \) or \( A(K^\circ) \) is not closed (the implication \( \Leftarrow \) in (16) is always valid, while for the implication \( \Rightarrow \) one can give counterexamples).

Note that the Farkas lemma can be interpreted as an alternative theorem. Having in view this remark, in the literature appeared quite many results called generalized Farkas lemma (see [10] for a survey); however, in our opinion, this is an exaggeration because any time when we write \( X = E \cup (X \setminus E) \) for some set \( E \) we should have a generalized Farkas lemma. We consider that a proper generalization of the Farkas lemma is when in the description of \( E \) or \( X \setminus E \) instead of linear functions and operators one has sublinear functions or operators; this is because a very important use of the Farkas lemma is in deriving necessary optimality conditions in smooth optimization, and now in nonsmooth analysis (and optimization) the derivative is replaced generally by a certain directional derivative (which generally is sublinear).

The second interpretation of the Farkas lemma is as a solvability result, more precisely the solvability of the equation \( Ay = b \) with \( y \geq 0 \), or something similar. Seen like this, the goal of several papers was to give characterizations for the solvability of the previous equation. The discussion above shows that when the order cone is not polyhedral, generally the classic characterization does not hold; however see [16, Th. 19] for a Farkas lemma type result related to semidefinite programming.

In the sequel we shall deduce the solvability of the equation \( Ay = b \) with \( y \geq 0 \) using (15). Because for \( b = 0 \) it is clear that the equation \( Ay = b \) with \( y \geq 0 \) is feasible (and (16) holds), in the sequel one assumes \( b \neq 0 \). The next result was obtained in [12, Th. 5.1] in the framework of Banach spaces and, in this context, the element \( x_0 \) is taken in int \( K^\circ \) (which, of course, equals \( K^\# \) in this case).

**Proposition 4.1** Assume that \( b \neq 0 \). Then \( Ay = b, y \geq 0 \) has solution if and only if there exist a nonempty compact convex set \( C \) with \( 0 \notin C \subset K^\circ, \delta > 0 \) and \( x_0 \in P^\# \) such that \( A^*z + x_0 \in P^\circ \Rightarrow \langle z, b \rangle \geq -\delta, \) where \( P := \mathbb{R}_+C \).

**Proof.** Assume that \( y_0 \geq 0 \) is such that \( Ay_0 = b \). Take \( C = \{y_0\} \). Because \( b \neq 0 \), \( C \) satisfies the desired conditions. Take also \( x_0 \in X \) such that \( \langle x_0, y_0 \rangle = 1 \) and \( \delta := 1 \). Then \( x_0 \in P^\# \) and \( P^\circ = \{x \in X : \langle x, y_0 \rangle \geq 0\} \). If \( A^*z + x_0 \in P^\circ \) then \( 0 \leq \langle A^*z + x_0, y_0 \rangle = 1 + \langle z, b \rangle, \) whence \( \langle z, b \rangle \geq -\delta \).

Conversely, assume that \( C, P, x_0 \) and \( \delta \) are as in the statement. We can apply [21, Th. 3], or, as in the proof of Proposition 2.1, because 0 is a solution of \( A^*z + x_0 \in P^\circ \), we have
\[
[A^*z + x_0 \in P^\circ \Rightarrow \langle z, b \rangle \geq -\delta] \Leftrightarrow \langle t, z \rangle \in \mathbb{R}_+ \times Z, A^*z + tx_0 \in P^\circ \Rightarrow \langle z, b \rangle + \delta t \geq 0
\]
\[
\Leftrightarrow \varphi \in \{(\mathbb{R}_+ \times Z) \cap T^{-1}(P^\circ)\}^\circ = cl\{(\mathbb{R}_+ \times \{0\}) + T^*(P^{\circ^\circ})\},
\]
where \( T : \mathbb{R} \times Z \rightarrow X, T(t, z) := A^*z + tx_0 \) and \( \varphi : \mathbb{R} \times Z \rightarrow \mathbb{R}, \varphi(t, z) := \langle z, b \rangle + \delta t \). Then \( T^*(y) = \langle (x_0, y), Ay \rangle \) for \( y \in Y \) and \( P^{\circ^\circ} = P \) because \( P \) is a closed convex cone. Hence there exist the nets \( (s_i)_{i \in I}, (t_i)_{i \in I} \subset \mathbb{R}_+, (c_i) \subset C \) such that \( s_i + t_i \langle x_0, c_i \rangle \rightarrow \delta, t_i \rightarrow 0 \) and \( s_i \rightarrow s, t_i \rightarrow t \) with \( s, t \in [0, \infty] \). Because \( x_0 \in P^\# \) we have that \( \langle x_0, c_i \rangle \rightarrow 0 \), and so \( s, t < \infty \). If \( t = 0 \) we obtain the contradiction \( b = 0 \). Hence \( A(tc) = b \) and \( tc \in P \subset K^\circ \), which shows that \( y = tc \) is the desired solution.

In the sequel, as in [6], consider \( F := \{A^*z | \langle z, b \rangle = 0\} \subset X \) and \( J = cl(K - F) \). Of course, \( J \) is a closed convex cone, and so \( ((A^*)^{-1}(J))^\circ = cl(A(J^\circ)) \).

**Lemma 4.1** Assume that \( b \neq 0 \). Then \( J^\circ = K^\circ \cap A^{-1}(Rb) \) and
\[
b \in A(K^\circ) \Leftrightarrow b \in A\left(K^\circ \cap A^{-1}(Rb)\right) \Leftrightarrow b \in cl\left(A\left(K^\circ \cap A^{-1}(Rb)\right)\right).
\]
(17)
PROOF. Because $F$ is a linear space we have
\[ y \in J^0 \iff [x \in K, u \in F \Rightarrow \langle x - u, y \rangle \geq 0] \iff [y \in K^\circ, \langle u, y \rangle = 0 \ \forall u \in F]. \]
But
\[ [(u, y) = 0 \ \forall u \in F] \Rightarrow [(z, b) = 0 \Rightarrow \langle A^* z, y \rangle = 0] \Rightarrow [(z, b) = 0 \Rightarrow \langle z, Ay \rangle = 0] \Rightarrow [\exists \lambda \in \mathbb{R} : Ay = \lambda b] \Rightarrow y \in A^{-1}(\mathbb{R}b). \]
Therefore, $J^0 = K^\circ \cap A^{-1}(\mathbb{R}b)$. The first equivalence in (17) is obvious, as well as the implication $\Rightarrow$ in the second equivalence. Let $b \in \text{cl} (A (K^\circ \cap A^{-1}(\mathbb{R}b)))$. Then there exists a net $(y_i)_{i \in I} \subset K^\circ \cap A^{-1}(\mathbb{R}b)$ such that $Ay_i \to b$. Since $y_i \in A^{-1}(\mathbb{R}b)$ we have that $Ay_i = \beta_i b$ for some $\beta_i \in \mathbb{R}$. Hence $\beta_i b \to b$. Since $b \neq 0$ we have that $\beta_i \to 1$. Hence $\beta_i > 0$ for some $i$, and so $b = A(\beta_i^{-1} y_i) \in K^\circ$, whence $b \in A(K^\circ)$. \qed

Using Lemma 4.1 we obtain the following novel characterization of the solvability of the equation $Ay = b, y \geq 0$.

COROLLARY 4.1 Let $z_0 \in Z$ be such that $\langle z_0, b \rangle > 0$. Then $b \in A(K^\circ)$ if and only if $A^* z_0 \notin (-J)$.

PROOF. Taking into account the expression of $J^0$ in Lemma 4.1 we get
\[ A^* z_0 \notin (-J) \iff -A^* z_0 \notin (J^0) \iff \exists y_0 \in J^0 : \langle -A^* z_0, y_0 \rangle < 0 \]
\[ \iff \exists y_0 \in K^\circ, \beta \in \mathbb{R} : Ay_0 = \beta b, \langle A^* z_0, y_0 \rangle = \beta \langle z_0, b \rangle > 0 \]
\[ \iff [\exists y_0 \in K^\circ, \beta > 0 : Ay_0 = \beta b] \iff b \in A(K^\circ). \]

The proof is complete. \qed

Using again Lemma 4.1 we get the following result from [6] considered as being comparable to Lasserre's results in [12] (that is, comparable to Proposition 4.1).

PROPOSITION 4.2 ([6, Th. 2]) One has
\[ [\exists y \in K^\circ : Ay = b] \iff [A^* z \geq_J 0 \Rightarrow \langle z, b \rangle \geq 0]. \]

PROOF. Indeed, using (15) with $K$ replaced by $J$ and Lemma 4.1 we obtain
\[ [A^* z \geq_J 0 \Rightarrow \langle z, b \rangle \geq 0] \iff b \in ((A^*)^{-1}(J))^\circ \iff b \in \text{cl} (A(J^0)) \iff b \in A(K^\circ), \]
which ends the proof. \qed

In [6] the condition $A^* z \geq 0 \Rightarrow \langle z, b \rangle \geq 0$ is denoted by FC (Farkas condition) and the condition $A^* z \geq_J 0 \Rightarrow \langle z, b \rangle \geq 0$ is denoted by GFC (generalized Farkas condition). It is clear that GFC $\Rightarrow$ FC. Hence, from Corollary 4.1 one gets immediately ([6, Lem. 4]) which asserts that GFC holds if and only if FC holds and $A^* z_0 \notin (-J)$.

We conclude with the next result obtained in [6, Lem. 3] for dim $X < \infty$ and $K$ polyhedral.

PROPOSITION 4.3 If $K - F$ is closed then FC and GFC are equivalent.

PROOF. As observed above GFC $\Rightarrow$ FC always. Assume that $K - F$ is closed and FC holds. Take $z$ such that $A^* z \in J = \text{cl}(K - F) = K - F$. Then $A^* z = z - A^* z'$ with $x \in K$ and $z' \in Z$ such that $\langle z', b \rangle = 0$. So, $A^* (z + z') = x \in K$, whence, by FC, $0 \leq \langle z + z', b \rangle = \langle z, b \rangle$. \qed

5. Appendix. One of the referees suggested new proofs, using Convex analysis, for Propositions 2.1–2.3, as well as alternative formulations for conditions $A.3'$ and $B.3'$; the new proof of Proposition 2.3 shows that conditions $A.2'$ and $A.3$ are superfluous for obtaining its conclusion. We present them below. As in Sections 1 and 2, $X$ and $Y$ are real linear spaces in separated duality.

A1) Condition $A.3'$ is equivalent to $K_0 := \{ x \in L \cap K \mid \langle x, \overline{y} \rangle = 0 \} \subset -K_0$. Since $K_0$ is a convex cone, this is equivalent to the fact that $K_0$ is a linear subspace. Similarly, $B.3'$ is equivalent to the fact that $\{ y \in L^+ \cap K^\circ \mid \langle \overline{\lambda}, y \rangle = 0 \}$ is a linear subspace of $Y$. \qed
A2) Let $A \subset X$ be a nonempty convex set. The quasi-relative interior of $A$ is the set $\text{qri}A := \{a \in A \mid \text{cl}(\mathbb{R}^+(A - a)) \text{ is a linear space}\}$ (see [2], [22]). It is clear that $\text{cl}(\mathbb{R}^+(A - a))$ is a linear space iff $(A - a)^\circ$ is a linear subspace of $Y$. In fact, if $a \in X$ and $\text{cl}(\mathbb{R}^+(A - a))$ is a linear space then necessarily $a \in \text{cl}A$. Moreover, if $\dim X < \infty$ then $\text{icr}A = \text{qri}A$; see [2] or [22, Section 1.2] for results on the quasi-relative interior. Therefore, if $\dim X < \infty$, $A$ is a convex cone and $a \in \text{cl}A \setminus \text{icr}A$ then $(A - a)^\circ = \{x^* \in A^* \mid \langle a, x^* \rangle = 0\}$ is not a linear subspace of $X^*$.

A3) Condition A.1’ is equivalent to $\bar{y} \in K^o + L^\perp$. Moreover, $(K^o + L^\perp - \bar{y})^\circ = \{x \in L \cap \text{cl}K \mid \langle x, \bar{y} \rangle \leq 0\}$. Hence, if $\bar{y} \in K^o + L^\perp$ then $(K^o + L^\perp - \bar{y})^\circ = \{x \in L \cap \text{cl}K \mid \langle x, \bar{y} \rangle = 0\}$. Therefore, if A.1’ is satisfied and $K$ is closed then A.3’ holds iff $(K^o + L^\perp - \bar{y})^\circ$ is a linear space iff $\overline{y} \in \text{qri}(K^o + L^\perp)$; moreover, if $\dim X < \infty$ then A.3’ holds iff $\bar{y} \in \text{icr}(K^o + L^\perp)$. Said differently, if $\dim X < \infty$ and $K$ is closed then [A.1’ and A.3’] holds iff $\bar{y} \in \text{icr}(K^o + L^\perp)$ (= $L^\perp + \text{icr}K^o$). Similarly, if $\dim X < \infty$ and $K$ is closed then [A.1’ and A.3] holds iff $\bar{y} \in \text{int}(K^o + L^\perp)$.

A4) We denote by $\iota_A$ and $\sigma_A$ the indicator and support functions associated to $A \subset X$, respectively; for the other notations and results used below see [17] or [22]. Fix $\overline{y} \in Y$ and consider

$$
\phi_1, \phi_2 : X \to \mathbb{R}, \quad \phi_1(x) := \langle x, \overline{y} \rangle + \iota_L(x), \quad \phi_2 = \iota_{(-K)}.
$$

Then

$$
\phi_1^* = \iota_{(\overline{y} + L^\perp)}, \quad \phi_2^* = \iota_{K^o}, \quad \phi_1^* + \phi_2^* = \iota_{(\overline{y} + L^\perp) \cap K^o}.
$$

In the sequel we assume that A.1’ holds, that is, $\overline{y} \in K^o + L^\perp$. Then $\alpha := \sigma_{(\overline{y} + L^\perp) \cap K^o}$ is a proper lower semicontinuous (lsc) sublinear function on $X$ and $\alpha = (\phi_1^* + \phi_2^*)^\circ$, or equivalently $\phi_1^* + \phi_2^* = \alpha^\circ$. Moreover, setting $\beta := \phi_1 \circ \phi_2$, we have that $\beta$ is a convex function and $\beta^* = \phi_1^* + \phi_2^* = \alpha^\circ$. (In fact $\alpha(\overline{y})$ and $\beta(\overline{y})$ are exactly the values of the problems (3) and (4), respectively.) Since $\alpha$ is a proper lsc convex function we have that $\alpha = \beta^\circ = \overline{\beta}$, where $\overline{\beta}$ is the lsc envelope of $\beta$. Because $\text{epi}_\alpha \beta = \text{epi}_\alpha \phi_1 + \text{epi}_\alpha \phi_2$, we deduce that

$$
\text{epi} \alpha = \text{cl}(\text{epi} \phi_1 + \text{epi} \phi_2) = \text{cl}(\mathbb{R}^+_x \setminus (-K) + \{(\langle x, \overline{y} \rangle, x) \mid x \in L\});
$$

the conclusion of Proposition 2.1 follows.

A5) Assume that $\dim X < \infty$, $K$ is closed and [A.1’, A.3’] holds. As seen in A3) we have that $\overline{y} \in \text{icr}(K^o + L^\perp)$, or equivalently 0 $\in \text{icr}(\text{dom} \phi_1^* - \text{dom} \phi_2^*)$. By [17, Th. 16.4, Cor. 6.6.2] or [22, Th. 2.8.4(viii)] we obtain that $(\phi_1^* + \phi_2^*)^\circ = \phi_1^* \circ \phi_2^*$ and the convolution is exact, that is, $\alpha = \phi_1 \circ \phi_2 = \beta$ and the infimum in the definition of $\beta$ is attained (when finite). This is the conclusion of Proposition 2.2.

A6) If $\dim X < \infty$ we have that $\beta(x) = \overline{\beta}(x)$ for all $x \in X \setminus \overline{\text{dom} \beta \setminus \text{icr}(\text{dom} \beta)}$. But $\text{dom} \beta = \text{dom} \phi_1 + \text{dom} \phi_2 = L - K$ is a convex cone and $(\text{dom} \beta)^\circ = L^\perp \cap (-K^o)$. If $\overline{y}$ verifies B.3’ then by A2) we have that $\overline{y} \in X \setminus \overline{\text{dom} \beta \setminus \text{icr}(\text{dom} \beta)}$. Hence $\alpha(\overline{y}) = \beta(\overline{y})$. This proves that the conclusion of Proposition 2.3 holds without conditions A.2’ and A.3.

A7) Observe that the primal problem (3) is equivalent to problem $(P) : \text{minimize } F(y, 0) \text{ s.t. } y \in Y$, where $F(y, z) := \iota_{(\overline{y} + L^\perp)}(y + z) + \iota_{K^o}(y - \overline{\beta}, y)$; it is clear that $F$ is a proper (under condition A.1’) and lsc convex function. When $\dim X < \infty$, the problem $(P)$ is strongly consistent (as defined in [17, Section 29]) iff $\overline{y} \in \text{icr}(K^o + L^\perp)$, and $(P)$ is strictly consistent iff $\overline{y} \in \text{int}(K^o + L^\perp)$. This shows that when $\dim X < \infty$ and $K$ is closed Proposition 2.2 follows using [17, Th. 30.4].

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