A critical view on invexity

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Abstract: The aim of this note is to emphasize the fact that in many papers on invexity published in prestigious journals there are not clear definitions, trivial or not clear statements and wrong proofs. We also point out the unprofessional way of answering readers' questions by some authors. We think that this is caused mainly by the lack of criticism of the invexity community
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1 Introduction

There are a lot of articles written on invexity. A search in MathScieNet gives 350 articles having in their titles the words "invex, invexity, preinvex" (310 in Zentralblatt für Mathematik).

In my opinion, the number of articles devoted to invexity (invex and generalized invex functions) is too big with respect to its importance. In fact, when I read the first time about (classic!?) invex functions, in the differentiable case, I realized that this is another way of saying that any critical (or stationary) point of the function, that is, a point at which the differential is 0, is necessarily a global minimum. Then the notion was generalized to non differentiable functions, but saying the same thing: every critical point (in the sense that 0 is in a certain type of subdifferential at that point) is a global minimum. If one looks to the applications of the results in Ref. 1 we see that they state something like: every local solution is a global one.

Rapidly one had generalizations of the notion: quasiinvex functions, preinvex functions, and so on. The common feature for many papers on invexity and its generalizations is the lack of clarity of the notions and statements of the results and doubtful proofs; when the proofs are correct many of them are trivial. After the first draft of this note was written (and after the change of messages with some authors of the cited papers) I had a closer look to several reviews in Mathematical Reviews and Zentralblatt für Mathematik (referred in Section 4), some of them related to the quoted articles; it seems that the opinions of the reviewers didn't influence the authors of papers on invexity.

2 About statements and proofs

Let us consider the following text quoted from Ref. 1 (Ref. 1 appears as being cited 37 times in Google Scholar and 4 times in MathSciNet at the time of writing this note):

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"**Definition 1.1**. See Refs. 1–2. A set $K \subseteq \mathbb{R}^n$ is said to be invex if there exists a vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$x, y \in K, \ \lambda \in [0, 1] \Rightarrow y + \lambda \eta(x, y) \in K.$$

Remark 1.1. A convex set is an invex set; i.e., take $\eta(x, y) = x - y$. But the converse does not hold."[1]

Note on this remark: Of course, the converse does not hold because by Definition 1.1 in Ref. 1 quoted above, any set is invex: just take $\eta(x, y) := 0$. The honest definition is: let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a function. The set $K \subset \mathbb{R}^n$ is said to be η-invex if ...

Or there is another formulation: One says that f is (θ, α)-d invex if there exist θ etc such that f satisfies a certain condition involving θ etc. Of course, correctly is to introduce first θ, α, d and after that to say that f is (θ, α)-d invex if ...

Let us quote from Ref. 2:

"**Theorem 8.** A function $f : R^n \to R$ is B-$(0, r)$-invex (B-$(0, r)$-incave) with respect to η and b on R^n if and only if its every stationary point is a global minimum (maximum) in R^n."

At least two remarks are in order with respect to this statement. The first one: If the statement is true, f is B-$(0, r)$-invex with respect to η and b (in the sense of Definition 1 in Ref. 2 if and only if f is invex (because, as seen above, the invexity of the differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is equivalent to the fact that every stationary point is a global minimum); so which is the need to introduce B-$(0, r)$-invexity? The second one: The statement gives the impression that the functions η and b, as well as $r \in \mathbb{R}$, are given. Consider $n = 1$, $r = 0$, $b(x, u) := 1$ and $\eta(x, u) := u^{-1} \left(x^2 - u^2 \right) + \operatorname{sgn} u$ for $u \neq 0$, $\eta(x, u) := 0$ for $u = 0$. Taking $f(x) = \frac{1}{2} x^2$ for $x \in \mathbb{R}$ we see that every stationary point of f (that is, $u = 0$) is a global minimum, but f is not B-$(0, r)$-invex with respect to η and b on $\mathbb{R}$. Let us quote also from Definition 8 in Ref. 3:

"Let $S \subset R^n$ be a nonempty invex set with respect to η. A function $f : S \to R$ is said to be pre-invex with respect to η if, there exists a vector-valued function $\eta : S \times S \to R^n$ such that the relation ...";

Definition 9 in Ref. 3 is obtained by changing pre-invex by invex. So, first η is given, and one line below one asks the existence of an η.

I am not used with this kind of text in mathematics. If invexity is not a domain of mathematics it is advisable to say it explicitly. Why is this important? Because we are judged in comparison with other mathematicians for getting jobs, for promotions, for getting grants. Even if one declares that invexity is not a domain of mathematics, this does not change a lot the situation because now one asks for interdisciplinarity.

Why did I ask if invexity is a topic in mathematics? Because I had the impression that the next text quoted from Ref. 4 is not a mathematical text (Ref. 4 appears as being cited 21 times in Google Scholar and 3 times in MathSciNet):

"**Remark 2.3.** We will show that Assumption C holds if
$\eta(x, y) = x - y + o(x - y)$. In fact, the following two equalities hold:
(i) $\eta(y, y + \lambda \eta(x, y))$

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[1] To see the precise coordinates of the references referred in the quoted texts one might consult the reviews on MathSciNet of the corresponding articles listed at the end of this note.

$$= \eta(y, y + \lambda(x - y + o(x - y)))$$
$$= -\lambda(x - y + o(x - y)) + o(\lambda(x - y + o(x - y)))$$
$$= -\lambda[x - y + o(x - y) + o(x - y + o(x - y))]$$
$$= -\lambda[x - y + o(x - y)]$$
$$= -\lambda\eta(x, y);\text{"} \ ^2$$

(I didn't quote the second equality which is of the same type).

Seeing this I sent messages to the authors of Ref. 4 asking:

"What do you mean by $\eta(x, y) = x - y + o\left(\|x - y\|\right)$ in Remark 2.3?",

but the answers were unsatisfactory. I don't cite the answers here being contained in private correspondence.

Which is this Condition C (or Assumption C)?
I quote again from page 610 of Ref. 4 (see also page 116 of Ref. 5):

"**Assumption C.** See Ref.6. Let $\eta : X \times X \to \mathbb{R}^n$. Then, for any $x, y \in \mathbb{R}^n$ and for any $\lambda \in [0, 1]$,
$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y)$, $\eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y)$."

Somewhere it is written that $X \subset \mathbb{R}^n$; probably for the authors it is not very important to speak about $\eta(x, y)$ when x or y is not in X. Let us consider $X = \mathbb{R}^n$. (Also note that in Definition 2.4 of Ref. 4 $\eta : X \times X \to \mathbb{R}$, that is, η takes its values in $\mathbb{R}$ instead of $\mathbb{R}^n$.)

To see that the condition $\eta(x, y) = x - y + o(x - y)$ does not imply Condition C let us consider $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $\eta(x, y) = x - y + (x - y)^2$. Of course η satisfies the hypothesis of the statement in Remark 2.3 of Ref. 4 (above). For $\lambda = 1$, the first relation of Condition C is equivalent to each of the next ones: $-\eta(x, y) + (\eta(x, y))^2 = -\eta(x, y)$, $(\eta(x, y))^2 = 0$, $\eta(x, y) = 0$, $(x - y)(1 + x - y) = 0$, $x - y \in \{0, -1\}$. So, taking $x = 0$, $y = 2$ and $\lambda = 1$ one sees that η does not verify Condition C.

In Ref. 5 one finds:

"**Example 2.1**. Let
$$\eta(x, y) = \begin{cases} x - y & \text{if} \quad x \geq 0, \ y \geq 0; \\ x - y & \text{if} \quad x \leq 0, \ y \leq 0; \\ -2 - y & \text{if} \quad x > 0, \ y \leq 0; \\ 2 - y & \text{if} \quad x \leq 0, \ y > 0. \end{cases}$$
Then, it is easy to verify that η satisfies Condition C."

A similar example is the following quoted from Ref. 4 (which is very close to that quoted above from Ref. 5, as well as to Example 1.1 in Ref. 1):

"**Example 2.2**. Let
$f(x) = -|x|$, $\forall x \in K = [-2, 2]$, and let
$$\eta(x, y) = \begin{cases} x - y & \text{if} \quad x \geq 0, \ y \geq 0; \\ x - y & \text{if} \quad x < 0, \ y < 0; \\ -2 - y & \text{if} \quad x > 0, \ y \leq 0; \\ 2 - y & \text{if} \quad x \leq 0, \ y > 0. \end{cases}$$

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[2]Related to this piece of non-mathematics let us quote from what the authors say in the first footnote on the first page of Ref. 4: "The authors are thankful to ..., and **three anonymous referees** for their many valuable comments on an early version of this paper. The authors are also grateful to Professor B.D. Craven for some discussion on this paper". (C.Z.' emphasis.)

Then, it is easy to verify that f is invex with respect to η on K and that f and η satisfy Assumptions A and C. However, f is not convex."

The authors seem to not realize that η defined in these two examples is not a function because $\eta(2,0)$ gives 2 using the first expression and -2 using the third expression. A possible modification for η defined in Example 2.1 of Ref. 5 could be:

$$\eta(x,y) = \begin{cases} x - y & \text{if} \quad xy \geq 0, \\ 2 - y & \text{if} \quad xy < 0. \end{cases}$$

Take $x, y \in \mathbb{R}$ with $x > 0 > y$ and $\lambda \in [0,1]$ and let us look to the second relation in Condition C. We have that $\eta(x,y) = 2 - y$ and $y' := y + \lambda\eta(x,y) = y + \lambda(2 - y)$. Assuming that $y' \geq 0$, then $\eta(x, y') = x - y' = x - 2 + (1 - \lambda)\eta(x,y)$. Hence, in such a situation ($y < 0$ and $y + \lambda(2 - y) \geq 0$) one has $\eta(x, y + \lambda\eta(x,y)) = (1 - \lambda)\eta(x,y)$ if and only if $x = 2$. Is it possible to have $y < 0$, $y + \lambda(2 - y) \geq 0$ and $\lambda \in [0,1]$? The answer is YES! Just take $\lambda = 1$. Hence, for $x = 1$, $y = -1$ and $\lambda = 1$ the second relation in Condition C is not verified because $\eta(1, -1) = 3$, $\eta(1, -1 + 3) = \eta(1, 2) = -1 \neq 0$.

In fact an adequate modification of the function η in Example 2.1 from Ref. 5 (or Example 2.2 in Ref. 4) is

$$\eta(x,y) = \begin{cases} x - y & \text{if} \quad xy \geq 0, \\ 2 - y & \text{if} \quad x < 0, \ y > 0, \\ -2 - y & \text{if} \quad x > 0, \ y < 0. \end{cases}$$

The function η defined in this way satisfies indeed Condition C.

Somewhere (say [S]) it was said that

$$\eta(y + \lambda_2\eta(x,y), y + \lambda_1\eta(x,y)) = (\lambda_2 - \lambda_1)\eta(x,y) \quad \forall x, y \in \mathbb{R}^n, \ \forall \lambda_1, \lambda_2 \in [0,1], \qquad (1)$$

whenever η verifies Condition C; and for this the proof of Theorem 3.1 in Ref. 4 was cited. In fact I was determined by [S] to look at Ref. 4 and Ref. 5. Of course, relation (1) is nice and good to have; moreover, for $\lambda_2 = 0$ one recovers the first relation in Condition C.

Looking at the proof of Theorem 3.1 in Ref. 4 (but one can look also at the proof of Theorem 2.1 in Ref. 4 for the same text), one observes that one takes $0 < \lambda_2 < \lambda_1 < 1$ and one obtains relation (14) of Ref. 4 I am quoting below:

"$\eta(y + \lambda_1\eta(x,y), y + \lambda_2\eta(x,y))$
$= \eta(y + \lambda_1\eta(x,y), y + \lambda_1\eta(x,y) - (\lambda_1 - \lambda_2)\eta(x,y))$
$= \eta(y + \lambda_1\eta(x,y), y + \lambda_1\eta(x,y) + \eta(y, y + (\lambda_1 - \lambda_2)\eta(x,y)))$
$= -\eta(y, y + (\lambda_1 - \lambda_2\eta(x,y))$
$= (\lambda_1 - \lambda_2)\eta(x,y)$."

The first equality is obvious, the second as well as the fourth follow from the first relation in Condition C [however it is $= -\eta(y, y + (\lambda_1 - \lambda_2)\eta(x,y))$ instead of $= -\eta(y, y + (\lambda_1 - \lambda_2\eta(x,y))$]. What is used to obtain the third equality? Setting $y' := y + \lambda_1\eta(x,y)$, the expression on the third line becomes $\eta(y', y' + \eta(y, y + (\lambda_1 - \lambda_2)\eta(x,y)))$. In order to get the expression on the fourth line (using directly Condition C) we should have $\eta(y'', y'' + \eta(y, y''))$ with $y'' := y + (\lambda_1 - \lambda_2)\eta(x,y)$. Is $y' = y''$? In fact $y' = y''$ if and only if $\eta(x,y) = 0$ or $\lambda_2 = 0$.

Maybe (1) is true whenever Condition C holds, but some additional arguments must be provided.

I do not propose myself to mention all doubtful sentences or statements in articles about invexity, but the majority I had occasion to browse are like that.

3 About the triviality of results and generalizations

Another problem with invexity is given by the triviality of some results or generalizations. Let us mention some of them found in recent articles published in prestigious journals.

It is well known that for a Gâteaux differentiable function $f : D \to \mathbb{R}$ with D an open subset of a normed vector space (but X could be a topological vector space), for any (distinct) points $a, b \in D$ with $[a, b] := \{\lambda a + (1 - \lambda)b \mid \lambda \in [0, 1]\} \subset D$, there exists $c \in]a, b[:= \{\lambda a + (1 - \lambda)b \mid \lambda \in (0, 1)\}$ such that $f(b) - f(a) = \nabla f(c)(b - a)$ (the proof being immediate using the real-valued function φ defined by $\varphi(t) := f(ta + (1 - t)b)$ for those $t \in \mathbb{R}$ with $ta + (1 - t)b \in D$). Which are the main results obtained in Ref. 3? I don't speak about Theorems 11 and 12 which are just rewriting of the definitions of convexity and pre-invexity (in Theorem 11 of Ref. 3 no need of differentiability of f). Let us quote Theorem 14 in Ref. 3:

"**Theorem 14.** Let $S \subset R^n$ be a nonempty invex set with respect to $\eta : S \times S \to R^n$, and P_{ab} be an arbitrary η-path contained in int S. Moreover, we assume that $f : S \to R$ is defined on S and differentiable on int S. Then, for any $a, b \in S$, there exists $c \in P_{ab}^0$ such that the following relation

$$f(a + \eta(b, a)) - f(a) = [\eta(b, a)]^T \nabla f(c) \qquad (8)$$

holds."

Because (c.f. Definition 5 in Ref. 3) $P_{ab} := [a, a + \eta(a, b)]$ and $P_{ab}^0 :=]a, a + \eta(a, b)[$, we see that Theorem 14 in Ref. 3 is an immediate consequence of the usual mean-value theorem mentioned above. (Note that it is not said what kind of differentiability is asked for f — Gâteaux or Fréchet.) Probably the next paper will deal with such a result in infinite dimensional spaces, then with α-η invex functions (mentioned below). Theorem 17 in Ref. 3 deals with a Taylor's expansion (of order 2) for f. Other "important" results (Theorems 21, 22) are immediate consequences of known results for derivable functions of one real variable. They could constitute easy exercises for students following a first course in analysis.

Another example in this sense is provided by Ref. 6. As seen in the title of Ref. 6, there is some G there. What is it? It is a function defined on a certain set $A \subset \mathbb{R}$ with values in $\mathbb{R}$ which is increasing ($s, t \in A$, $s < t \Rightarrow G(s) < G(t)$), and moreover G is differentiable. In fact G is defined on the image of a real-valued function f defined in its turn on an η-invex set $X \subset \mathbb{R}^n$. (By the way, if $A = I_f(X)$ is $\{0, 1\}$, what does differentiability of G mean?) One defines G-invex and G-pre-invex functions. Let us quote Definition 3 in Ref. 6:

"**Definition 3.** Let X be a nonempty invex (with respect to η) subset of R^n and $f : X \to R$ be a differentiable function defined on X. Further, we assume that there exists a differentiable real-valued increasing function $G : I_f(X) \to R$. Then f is said to be (strictly) G-invex at $u \in X$ on X with respect to η if there exists a vector-valued function $\eta : X \times X \to R^n$ such that, for all $x \in X$ ($x \neq u$), $G(f(x)) - G(f(u)) \geq G'(f(u))\nabla f(u)\eta(x, u)$ $(>)$.   (2)

If (2) is satisfied for any $u \in X$ then f is G-invex on X with respect to η."

Taking into account that for f (Fréchet) differentiable one has $\nabla h(u) = G'(f(u))\nabla f(u)$, where $h := G \circ f$, the inequality above says that $h(x) - h(u) \geq \nabla h(u)\eta(x, u)$. Having this inequality for all $x, u \in X$ this means that h is invex. So, one can say simply that f is G-invex (at u) if $G \circ f$ is invex (at u). This simple remark is not made in Ref. 6, but one has (quoted from Ref. 6):

"We remark that the G-invexity assumption generalizes a hypothesis of Avriel et al. [6], Avriel [7], Hanson [11] and Antczak [3] for differentiable functions. Thus, the following

remarks are true:

Remark 5. In the case when $\eta(x, u) = x - u$, we obtain a definition of a differentiable G-convex function introduced Avriel et al. [6].

Remark 6. Every invex function with respect to η introduced by Hanson [11] is G-invex with respect to the same function η, where $G : I_f(X) \to R$ is defined by $G(a) \equiv a$. The converse result is, in general, not true (see also Remark 13 and Example 14).

Remark 7. Every r-invex function with respect to η introduced by Antczak [1,3] is G-invex with respect to the same function η, where $G : I_f(X) \to R$ is defined by $G(a) = e^{ra}$, where r is any finite real number."

(However, note that for $r \leq 0$ the function G defined by $G(a) = e^{ra}$ is not increasing.)

It is suggestive to quote also the definition a G-pre-invex function (but probably the reader already guesses it):

"**Definition 9.** Let X be a nonempty invex (with respect to η) subset of R^n. A function $f : X \to R$ is said to be (strictly) G-pre-invex at u on X with respect to η if there exist a continuous real-valued increasing function $G : I_f(X) \to R$ and a vector-valued function $\eta : X \times X \to R^n$ such that for all $x \in X$ $(x \neq u)$,

$$f(u + \lambda\eta(x, u)) \leq G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(u))) \qquad (<). \qquad (3)$$

If (2) is satisfied for any $u \in X$ then f is G-pre-invex on X with respect to η."

Of course, the author does not (want to) observe that this means that $h := G \circ f$ is pre-invex at u. What does he obtain in Theorem 10 of Ref. 6? He obtains that f is G-invex provided f and G are differentiable and f is G-pre-invex. I quote from page 646 in Ref. 1:

"Recently, Pini (Ref. 6) showed that, if f is defined on an invex set $K \subseteq \mathbb{R}^n$ and if it is preinvex and differentiable, then f is also invex with respect to η; i.e., $f(y) - f(x) \geq \eta(y, x)^T \nabla f(x)$".

Of course, in Ref. 6 one gives a detailed proof. At page 646 of Ref. 1 one continues with:

"But the converse is not true in general. A counterexample was given in Ref. 6. However, Mohan and Neogy (Ref. 9) proved that a differentiable invex function is also preinvex under the following condition. **Condition C.** ..."

At page 1620 of Ref. 6 one says:

"The converse result is not true in general, that is, there exist G-invex functions with respect to η which are not G-pre-invex with respect to the same function η. To prove the converse theorem the function η should satisfy the following condition C (see [16]). **Condition C.** ..."

Of course one states Theorem 11 and one gives a detailed proof. As a conclusion for paper Ref. 6: The function f is G-"word" if $G \circ f$ is "word". If an existing result holds for "word" then in Ref. 6 one has a result for G-"word" with detailed proof. And this is published in a prestigious journal.

The case of Refs. 3, 6 is not singular. Let us have a look to Ref. 7 and its follower Ref. 8. Let us quote first from Ref. 7 two interesting phrases:

"In recent years, the concept of convexity has been generalized and extended in several directions using **novel and innovative** techniques"

and

"Motivated and inspired by the research going on in this **fascinating field**, we introduce a new class of generalized functions".

Let us quote again from page 698 of Ref. 7:

"Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle .,. \rangle$ and $\|.\|$ the inner product and norm respectively. Let $F : K \to H$ and $\eta(.,.) : K \times K \to R$ be continuous functions. Let $\alpha : K \times K \to R \setminus \{0\}$ be a bifunction. First of all, we recall the following well-known results and concepts.

Definition 2.1. Let $u \in K$. Then the set K is said to be α-invex at u with respect to $\eta(.,.)$ and $\alpha(.,.)$, if, for all $u, v \in K, t \in [0, 1]$, $u + t\alpha(v, u)\eta(v, u) \in K$. K is said to be an α-invex set with respect to η and α, if K is α-invex at each $u \in K$. The α-invex set K is also called $\alpha\eta$-connected set. Note that the convex set with $\alpha(v, u) = 1$ and $\eta(v, u) = v - u$ is an invex set, but the converse is not true."

First note that $u + t\alpha(v, u)\eta(v, u)$ above does not make sense if $H \neq \mathbb{R}$ because $u \in H$ and $t\alpha(v, u)\eta(v, u) \in \mathbb{R}$; next, if $\eta(.,.) : K \times K \to H$ (as in Ref. 8, then K is an α-invex set with respect to η iff K is η'-invex, where $\eta' := \alpha\eta$ (apparently not observed in Refs. 7, 8). Of course, in Definition 2.2 of Ref. 7 one says:

"The function F on the α-invex set K is said to be α-preinvex with respect to α and η, if $F(u + t\alpha(v, u)\eta(v, u)) \leq (1 - t)F(u) + tF(v), \ \forall u, v \in K, \ t \in [0, 1]$",

that is (I say), F is η'-preinvex (however, one must take $F : K \to R$ as in Ref. 8 instead of $F : K \to H$). In a similar way one obtains the corresponding definitions for "α-invex" replaced by "quasi α-preinvex" (see Definition 2.3 in Ref. 7, "logarithmic α-preinvex" (see Definition 2.4 in Ref. 7), "pseudo α-preinvex" (see Definition 2.5 in Ref. 7) from the definitions without α. (Note the interesting inequality $\max\{F(u), F(v)\} < \max\{F(u), F(v)\}$ from the displayed relation after Definition 2.4 in Ref. 7.) Maybe the next one is an exception:

"**Definition 2.6.** The differentiable function F on K is said to be an α-invex function with respect to α and η, if
$F(v) - F(u) \geq \langle \alpha(v, u)F'(u), \eta(v, u) \rangle$, $\forall u, v \in K$, where $F'(u)$ is the differential of F at $u \in K$. **The concepts of the α-invex and α-preinvex functions have played very important role in the development of convex programming; see [6,7].** Note that for $\alpha(v, u) = 1$, Definition 2.6 is mainly due to Hanson [1]".

Unfortunately not, even in this case, F is α-invex with respect to α and η iff F is η'-invex. What is new and surprising for me is the emphasized text above.

Similar remarks are valid for the notions of "$\alpha\eta$-monotone", "strictly $\alpha\eta$-monotone", "$\alpha\eta$-pseudomonotone", "quasi $\alpha\eta$-monotone", "strictly $\alpha\eta$-pseudomonotone" referred to an operator $T : K \to H$ (defined in Definition 2.7 in Ref. 7).

However, there are some notions which do not correspond to those for $\eta' := \alpha\eta$. These are those containing the word "strongly" in their definition: "strongly $\alpha\eta$-monotone" and "strongly $\alpha\eta$-pseudomonotone" operators (see Definition 2.7 in Ref. 7) as well as "strongly α-preinvex" (see Definition 2.8 in Ref. 7), "strongly α-invex" (see Definition 2.9 in Ref. 7), "strongly pseudo $\alpha\eta$-invex" (see Definition 2.10 in Ref. 7) and "strongly quasi α-invex" (see Definition 2.11 in Ref. 7) functions. The results which refer to these notions are Theorems 3.1–3.5 in Ref. 7. I do not propose myself to verify the correctness of these results (however see Example 6.1 in Ref. 8, but some of them probably are not true having in mind that Theorems

6.1 and 6.4 in Ref. 8 give alternative formulations for the sufficiency parts of Theorems 3.2 and 3.5 in Ref. 7, respectively. What I want to point out are the following facts:

1) If $\eta(u, u) = 0$ for some $u \in K$ then there do not exist pseudo α-preinvex, strictly α-invex and strictly pseudo α-invex functions with respect to α and η, as well as strictly $\alpha\eta$-monotone and strictly pseudo $\alpha\eta$-monotone operators. Note that if α and η satisfy Condition C at page 702 of Ref. 7 or condition (ii) in Theorem 6.1 of Ref. 8 then $\eta(u, u) = 0$ for every $u \in K$.

2) In some proofs of the statements in Refs. 7 and 8 one uses the relation $g(1) - g(0) = \int_0^1 g'(t)dt$, where g is a real-valued derivable function on a subset of $\mathbb{R}$ containing $[0, 1]$. In fact $g(t) := F(u + t\alpha(v, u)\eta(v, u))$ for $t \in [0, 1]$, where F is differentiable. It is a well-known fact that the formula $g(1) - g(0) = \int_0^1 g'(t)dt$ might not be true if g' is not Riemann integrable on $[0, 1]$. As an example take $g(t) := t^2 \sin t^{-2}$ for $t \in]0, 1]$, $g(0) := 0$.

3) In Theorems 6.1–6.4 of Ref. 8 one uses the condition "$\alpha(u, u + t\alpha(v, u)\eta(v, u)) = t\alpha(v, u)$, $\forall u, v \in K$, $t \in [0, 1]$". Taking $t = 0$, this implies that $\alpha(u, u) = 0$ for every $u \in K$, contradicting the assumption made before Definition 1.1 in Ref. 8 that α takes its values in $R \setminus \{0\}$. This shows that the domain of applicability of Theorems 6.1–6.4 in Ref. 8 is the empty set.

4   Conclusions

In this note we pointed out that several papers published in prestigious journals contain important drawbacks in the formulation of the notions and in the statements of the results, as well as very serious mistakes in the proofs. Also, there are many trivial generalizations of notions and results. In this sense it is useful to mention that there are several reviews in Mathematical Reviews and Zentralblatt für Mathematik which are concordant with our opinions; let us cite the reviews *MR1989930 (2004e:90091)* (for Ref. 4, by S. Komlosi) in which it is mentioned explicitly that Remark 2.3 of Ref. 4 is false by giving a counterexample; *Zbl 1094.26008 Noor, Muhammad Aslam On generalized preinvex functions and monotonicities. (English) [J] JIPAM, J. Inequal. Pure Appl. Math. 5, No. 4, Paper No. 110, 9 p., electronic only (2004). ISSN 1443-5756* (by J. E. Martínez-Legaz) in which it is mentioned that all the results in the paper follow from a simple observation; *Zbl 1096.26006 Noor, Muhammad Aslam; Noor, Khalida Inayat On strongly generalized preinvex functions. (English) [J] JIPAM, J. Inequal. Pure Appl. Math. 6, No. 4, Paper No. 102, 8 p., electronic only (2005). ISSN 1443-5756* (by J. E. Martínez-Legaz) in which, besides other remarks, it is mentioned a definition which does not make sense; *Zbl 1093.26006* (for Ref. 7, by N. Hadjisavvas) where it is mentioned that "Many other notions and properties introduced in this paper can be derived in the same way from the usual generalized invexity notions that can be found in other papers in the field. When this is not the case, mistakes occur frequently". We also pointed out the unprofessional way some authors answered questions related to their papers. In conclusion we consider that there are too many papers related to invexity, much more that the domain deserves. We consider that the editors of mathematical journals have to pay much more attention when accepting to publish such papers, taking into account at least the lack of criticism in the Invexity Community.

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